Princeton Plasma Physics Laboratory

PPPL- 4419

PPPL- 4419

DC-like Phase Space Manipulation and Particle Acceleration Using Chirped AC Fields

P.F. Schmit AND N.J. Fisch

June, 2009

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

DC-like phase space manipulation and particle acceleration using chirped AC fields

P. F. Schmit and N. J. Fisch
Princeton Plasma Physics Laboratory
Princeton, NJ 08543

Waves in plasmas can accelerate particles that are resonant with the wave. A DC electric field also accelerates particles, but without a resonance discrimination, which makes the acceleration mechanism profoundly different. We investigate the effect on a Hamiltonian distribution of an accelerating potential waveform, which could, for example, represent the average ponderomotive effect of two counterpropagating electromagnetic waves. In particular, we examine the apparent DC-like time-asymptotic response of the distribution in regimes where the potential structure is accelerated adiabatically. A highly resonant population within the distribution is always present, and we characterize its nonadiabatic response during wave-particle resonance using an integral method in the noninertial reference frame moving with the wave. Finally, we show that in the limit of infinitely slow acceleration of the wave, these highly resonant particles disappear and the response of the bulk distribution becomes identical to the response of a distribution to a uniform DC field.

PACS numbers:

I. INTRODUCTION

Through their work with autoresonant BGK modes in plasmas, Friedland *et al* uncovered an interesting phenomenon involving accelerated waves that has the apparent effect of shifting an entire particle distribution by a fixed displacement in velocity space, regardless of the initial configuration of the distribution [1, 2]. By adiabatically accelerating a sinusoidal wavetrain through a quasineutral particle distribution such that the initial and final phase velocities are far out of resonance with the electron distribution and on opposite sides of the distribution's boundaries in velocity space, the field induced in the plasma can become phase-locked with the drive field, creating a coherent accelerating wave structure. These waves can carry empty "holes" of phase space through the electron distribution, resulting in a net displacement of the entire electron distribution in the direction opposite to the acceleration of the wave. The effect was noted to have potential use as a current drive scheme in confined plasmas, and such an effect could also be used to develop high quality dense, monoenergetic particle beams. An even more fundamental physical notion underlies this phenomenon: it appears that under certain circumstances, a wave can produce an effect on a distribution indistinguishable from that of a uniform DC field. The consequence is that an AC field that time-averages to zero can potentially have an effect on a distribution identical to a DC field with non-vanishing time-average.

As it turns out, there are some caveats to this statement; in particular, for any accelerating wave scheme there exists a fraction of the initial distribution that is highly resonant with the accelerating wave and does not exhibit the adiabatic shift noted by Friedland. In a current drive or particle acceleration scheme, these highly resonant particles represent an inefficiency in the acceleration process, creating a counter-current in the case of a current drive scenario and a second beam population in the beam production scenario. This paper sets out to describe rigorously the full nature of the effect noted by Friedland *et al*, and in particular focuses on the behavior of these highly-resonant particles and shows that in a certain limit, their effects can be negated entirely. Furthermore, Friedland's adiabatic invariant model focuses primarily on the dynamics of the distribution boundary; consequently, we seek to prove explicitly that in certain limits, not only do the boundaries of the distribution obtain a DC-like constant impulse, but rather every particle throughout the distribution receives the same impulse.

One of the interesting features of a DC electric field is that it drives very efficiently electrical current, since during momentum transfer between the particles and the field, some particles lose energy and some particles gain energy. This feature is absent in most current drive schemes by waves, so that radiofrequency (RF) current drive methods generally are less efficient [3]. Therefore, to the extent that RF waves could possibly mimic the acceleration produced by DC fields, there may be opportunities for more efficient RF current drive.

The idea of using filled phase space holes to accelerate uniformly bunches of trapped particles originated with Sessler and Symon [4, 5]. Here, the accelerating wave was used to accelerate uniformly a small bucket of phase space, rather than the full distribution that we consider, with the aim of getting small beam spread. Hofmann made an explicit calculation for the acceleration of bunches of charges via the displacement of empty phase space buckets from the high energy side to the low energy side [6]. This effect is roughly the same effect observed by Friedland in the autoresonant BGK modes. In the case of free electron lasers [7], the phase space deceleration similarly makes use of a ponderomotive hole in phase space accelerated through a beam, so as to slow down the beam. The idea of producing holes in phase space through BGK modes was explored both experimentally and theoretically [8–10].

In the case of interest for us, instead of accelerating just one bucket of particles in phase space, our question is the extent to which we can accelerate the full distribution like a DC electric field. Here, not only is it of interest that each bit of phase space can be uniformly accelerated like a DC electric field would do, but also that the inevitable regions of phase space that undergo excess acceleration be kept vanishingly small.

To explore the extent to which waves can act like DC fields, we introduce what we call the τ-integral formalism in Section II. Then, in Section III, a simplified form of Friedland's results will be presented using the adiabatic invariant global model on a noninteracting Hamiltonian distribution in order to provide an intuitive perspective of the problem and motivate our rigorous result. Data from numerical simulations will be presented that lend credence to Friedland's model while simultaneously indicating its limitations and the need for a rigorous derivation to prove the apparent AC-DC correspondence. In Sections IV and V, the analytical solution for the peak resonant impulse delivered by an accelerating wave to a particle with arbitrary initial conditions will be derived, and in the limit of infinitely slow acceleration we will show that AC fields do indeed produce an effect on an arbitrary distribution of particles that is indistinguishable from that resulting from a momentary application of a uniform DC field. Finally, in sections VI and VII, we explore various means of particle acceleration utilizing waves and the potential benefits of using waves to mimic DC fields.

II. ACCELERATING POTENTIALS AND THE τ-INTEGRAL FORMALISM

This paper will examine the effect of a potential of the form

$$\Phi(x,t) = \Phi_0 \cos\left[k\left(x - \Psi(t)\right)\right], \tag{1}$$

on an arbitrary one-dimensional distribution of noninteracting classical particles. This potential is very similar to that used by Friedland *et al* as well as Shneider *et al* to approximate the effects of an accelerating beat wave, or optical lattice, generated by two counterpropagating frequency-chirped electromagnetic waves [1, 2, 11]. The salient features of the physical process are retained without requiring a self-interacting distribution with self-consistent fields, as would be needed in a plasma, which will help simplify the analysis. Friedland showed it is possible for an externally-driven plasma to produce a phase-locked self-field that retains the coherent structure of the original drive wave. Thus, the effect of the total field on the distribution is structurally similar to the effect of an external drive field on a noninteracting distribution [1, 2].

We begin by considering a general accelerating one-dimensional potential of the form $\phi\left(x - \psi\left(t\right)\right)$, with ψ an arbitrary function of time. An ideal reference frame in which to perform analysis would be the noninertial rest frame of the potential, whose coordinate is defined by $s \equiv x - \psi\left(t\right)$. In this coordinate system, the equation of motion becomes

$$m\ddot{x} = m\left(\ddot{s} + \ddot{\psi}\right) = -\frac{\partial \phi}{\partial x} = -\frac{\partial \phi}{\partial s}. \tag{2}$$

A conserved quasi-energy E can be identified in the case of constant acceleration of the potential, i.e. $\psi\left(t\right) = \frac{1}{2}at^2 + v_0 t + x_0$. The quasi-energy is found by multiplying Eq. 2 by $\dot{s}$:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{s}^2 + \phi(s) + m\ddot{\psi}s\right) \equiv \frac{dE}{dt} = ms\dddot{\psi}. \tag{3}$$

The term on the right hand side of Eq. 3 can be thought of as a driving term for the quasi-energy, but in the case of a potential accelerating at the constant rate a, we have $\dddot{\psi} = 0$, and so the quasi-energy is conserved. Thus, the dynamics arising from the accelerating potential are taken into account by the introduction of an additional linear potential $m\ddot{\psi}s = mas$ into the quasi-energy in the noninertial frame of reference moving with the potential.

We can calculate the impulse delivered to a classical particle by an accelerating potential by solving what we shall define as a τ-integral. We note that

$$\begin{aligned}\Delta\dot{x} &= \int_{t_i}^{t_f} \ddot{x}\, dt = \int_{t_i}^{t_f} \left(\ddot{s} + \ddot{\psi}\right) dt \\ &= \int_{t_i}^{t_f} \ddot{s}\, dt + \int_{s_i}^{s_f} \frac{\ddot{\psi}}{\dot{s}}\, ds \\ &= \dot{s}(t_f) - \dot{s}(t_i) + a\int_{s_i}^{s_f} \frac{ds}{\dot{s}},\end{aligned}$$

where we have switched to the coordinate representation $s \equiv x - \psi(t)$ and used the equation $\psi(t) \equiv 1/2at^2 + v_0 t + x_0$ for the constant acceleration case. We also used $s(t_i) \equiv s_i$ and $s(t_f) \equiv s_f$. In this reference frame, we know the quasi-energy defined in Eq. 3 is conserved, and so if $\dot{s}(t_i) = \dot{s}(s_i(t_i)) \equiv \dot{s}_i$ is specified by an initial condition, we can immediately determine $\dot{s}(t_f) = \dot{s}(s_f(t_f)) \equiv \dot{s}_f$. To simplify notation, we define the quantity τ as

$$\tau \equiv \int_{s_i}^{s_f} \frac{ds}{\dot{s}} + \frac{1}{a}\left(\dot{s}_f - \dot{s}_i\right), \tag{4}$$

with

$$\dot{s} = \left[\frac{2}{m}\left(E - \phi(s) - mas\right)\right]^{1/2},$$

and E is the conserved quasi-energy in the noninertial frame of reference. According to this definition,

$$\Delta\dot{x} = a\tau. \tag{5}$$

Thus, by calculating τ, defined in the noninertial reference frame, it is possible to determine the impulse delivered to a classical particle in the laboratory reference frame under the influence of a fixed-profile, accelerating potential over any interaction range. Time asymptotic impulses can be calculated by setting one bound of the integral at the classical turning point in the quasi-potential and the other at the infinity toward which the potential becomes increasingly negative, due to the presence of the extra linear term, and multiplying the result by two. This corresponds to the scenario where the wave starts with an infinite phase velocity in one direction and accelerates until it has an infinite phase velocity in the opposite direction.

III. THE CONSERVED ACTION MODEL AND ITS LIMITATIONS

The effect of the potential defined in Eq. 1 on a distribution of noninteracting particles in phase space in the limit of adiabatic, monotonic acceleration of the waveform can be determined approximately by a few simple arguments; note that this calculation follows the assumptions made by Friedland *et al* in [2]. By adiabatic acceleration we mean that the acceleration is small enough so that for most of the interaction period the change in the wave phase velocity during the time it take the particle to traverse one period of the wave is small compared to the relative velocity between the particle and the wave, v_{rel}, or in other words:

$$\frac{2\pi a}{k v_{rel}} \frac{1}{v_{rel}} \sim \frac{a}{k v_{rel}^2} \ll 1. \quad (6)$$

In the limit of adiabatic acceleration of the waveform we can examine the particle interaction with the wave over time scales comparable to $2\pi/kv_{rel}$, the particle transit time in the wave, during which time the wave phase velocity $\dot{\Psi}$ remains approximately constant. In this case, all phase space trajectories are given by the equation

$$\dot{x}_{\pm}(t) = \dot{\Psi}(t) \pm \left[\frac{2\Phi_0}{m}\left(\alpha - \cos\left[k\left(x - \Psi(t)\right)\right]\right)\right]^{1/2}, \quad (7)$$

where

$$\alpha \equiv \frac{1}{\Phi_0}\left[1/2m(\dot{x} - \dot{\Psi})^2 + \Phi_0 \cos[k(x - \Psi)]\right] \quad (8)$$

is a conserved quantity on this time scale and represents the ratio of the energy of a particle in the inertial rest frame of the wave to the maximum strength of the potential. Orbits with $\alpha > 1$ are topologically open, orbits with $\alpha < 1$ are topologically closed, and the orbit given by $\alpha = 1$ defines the separatrix dividing phase space into regions of purely untrapped versus purely trapped orbits. An example snapshot of the various phase space trajectories for transit-time scales is depicted in Fig. 1. Notice that trajectories for particles traveling at a velocity substantially different from the instantaneous velocity of the waveform, i.e. $\alpha \gg 1$, can be represented as approximately straight, fixed-velocity lines, since they are far out of resonance with the wave and hence barely feel its effects. This will be useful to us in the calculation that follows.

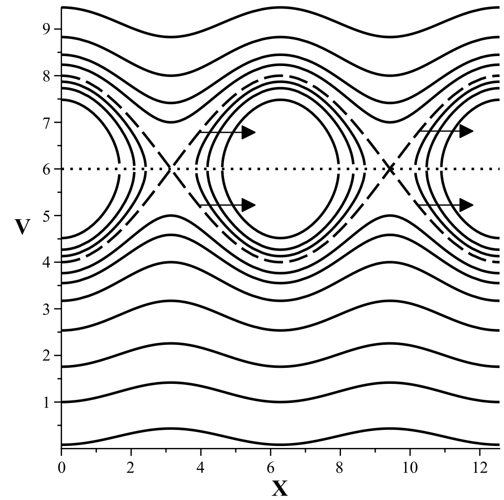


FIG. 1: Snapshot of phase space trajectories during transit-time scales, for which $\dot{\Psi} \approx$ constant. Separatrices are denoted by dashed line, instantaneous wave phase velocity $\dot{\Psi}$ is denoted by the dotted line, and the direction of translation of the orbits is indicated by the arrows.

We now envision a flat continuum particle distribution extending infinitely in one-dimensional configuration space but only across a finite range of velocity space, forming what is often called a "waterbag" distribution. When the wave is far out of resonance with the distribution, or in other words, when the wave phase velocity is substantially different from the velocity of any particle in the distribution, the waterbag is approximately unchanged in time. For example, if the distribution were rectangular, then it remains roughly rectangular, and the separatrices of the wave reside far outside the distribution. Thus all particle orbits are open, and the trapped-particle states confined within the wave's separatrices start out empty. We assume the wave is adiabatically accelerated, so that as the separatrices move across velocity space toward the distribution, the trapped particle states remain inaccessible to particles in the distribution, which all began on open orbits.

Now consider the situation where the wave has resonantly interacted with one velocity-space boundary of the distribution such that the separatrices now lie in the middle of the distribution. Because the waveform acceleration is presumed to be adiabatic, the closed-orbit trajectories remain unpopulated and the separatrices form the boundaries of empty holes inside the distribution. If the edges of the distribution in velocity space are far enough apart that neither is resonantly interacting with the wave, then they will both have the approximate asymptotic form of a straight line. We know that only the boundary of the distribution through which the separatrices came can be changed from the original picture, while the other boundary has yet to resonantly interact with the waveform. By Liouville's Theorem we must preserve the phase space volume of the distribution, and so the boundary that has already resonantly interacted with the wave must be displaced in order to account for the empty volume inserted into the distribution by the empty trapped-particle states. The action, or phase space volume $\int p\,dq$, per unit wavelength $\lambda = 2\pi/k$ of a straight line, which describes the non-resonantly interacting boundaries of the distribution, is given simply by $\mathrm{J}_{sl} = \lambda m\dot{x}$, while the phase space volume of the separatrices is found to be

$$\mathrm{J}_{sep} = \left[m \int_0^{\lambda} \left(\dot{x}_+ - \dot{x}_-\right)\, dx \right]_{\alpha=1} = 16\frac{\sqrt{m\Phi_0}}{k} \qquad (9)$$

(cf. Eq. 7). The conservation of action per unit wavelength of the distribution before and after the resonant interaction is stated simply as the jump condition

$$\mathrm{J}_{sl,f} = \mathrm{J}_{sl,i} + \mathrm{J}_{sep}, \qquad (10)$$

yielding the final result for the distribution boundary:

$$|\Delta\dot{x}| = \frac{8}{\pi}\sqrt{\frac{\Phi_0}{m}}, \qquad (11)$$

with the direction of the impulse opposite the direction of acceleration of the wave phase velocity. After the separatrices pass through the trailing boundary of the distribution and time-asymptotically travel off to infinity in velocity space, similar arguments reveal that the trailing boundary undergoes a shift in velocity space identical to the shift of the leading boundary, leaving another waterbag distribution of the same dimensions as the initial distribution, but displaced in velocity space. This is the result stated by Friedland *et al* [1, 2], and it led to his suggestion that this phenomenon could be used as a viable current drive scheme in plasmas.

However, this is not the whole story. Numerical simulations of particle motion reveal that for any small but nonzero acceleration, to be defined rigorously in the next section, the resulting impulse delivered to a distribution demonstrates a noticeable departure from the prediction calculated in Eq. 11, including the formation of tendrils of the distribution that extends as far in velocity space as the wave phase velocity is accelerated past resonance, opposite the direction of the adiabatic impulse, cf. Fig. 2. Note that due to the periodic symmetry of the potential in Eq. 1 under translations of $2\pi/k$ in x and the assumption of an infinitely extensive distribution in configuration space, we expect the patterns shown in Fig. 2 to be periodic across the distribution as well, and thus only one period of each pattern is shown.

Apparently, no matter how slow the acceleration, this highly resonant tendril never disappears completely. We will consider the structure of this nonadiabatically accelerated portion of the particle distribution in the next section.

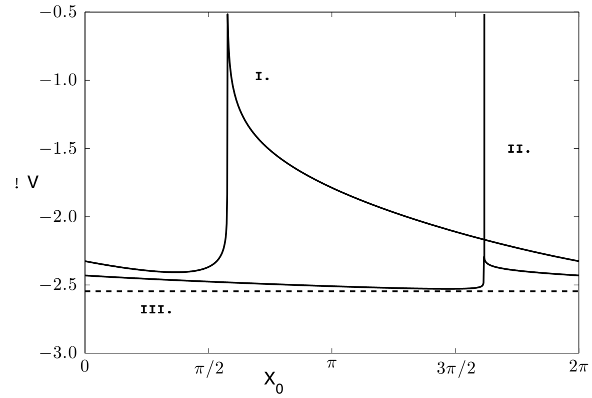


FIG. 2: Numerical simulations reveal the time-asymptotic impulse delivered to one period of the initial distribution for: I, $\epsilon = 10^{-1}$ and II, $\epsilon = 10^{-2}$, cf. Eq. 13. Line III shows the conserved action prediction for the impulse from Eq. 11.

IV. PHASE SPACE FIDELITY AND THE RESONANT IMPULSE

It is easy to see that parts of the distribution will not demonstrate the simple jump condition predicted in Eq. 11. We can rewrite the nonlinear equation of motion for a particle in the potential given by Eq. 1 in terms of a phase variable η, given by $\eta \equiv k\,(x - \psi(t))$:

$$\ddot{\eta} = \tilde{\phi}\,[\sin\eta - \epsilon] = -\frac{df}{d\eta}, \qquad (12)$$

with

$$\begin{aligned} \tilde{\phi} &\equiv a_{crit}k, \\ \epsilon &\equiv a/a_{crit}, \\ a_{crit} &\equiv \frac{\Phi_0 k}{m}. \end{aligned} \qquad (13)$$

The quasi-potential with the added linear term represented in dimensionless form is given by:

$$f(\eta) = \tilde{\phi}\,(\cos\eta + \epsilon\eta) \qquad (14)$$

and is plotted for $\tilde{\phi} = 1$ and several values of ϵ in Fig. 3.

Notice in Fig. 3 that any time the wave is accelerated such that $\epsilon \leq 1$, points of unstable equilibrium arise in the quasi-potential, i.e. $df/d\eta = 0$ and $d^2f/d\eta^2 < 0$, representing energies for which particles will resonantly interact with the wave for as long as the wave exists. In the language of the τ-integral formalism, particles with classical turning points at these points of the quasi-potential have time-asymptotic values $a\tau = \infty$ and are accelerated along with the wave indefinitely. In a situation where it is desirable to impart as much momentum to the distribution in the direction of the adiabatic bulk impulse as possible, these highly-resonant particles for which $a\tau$

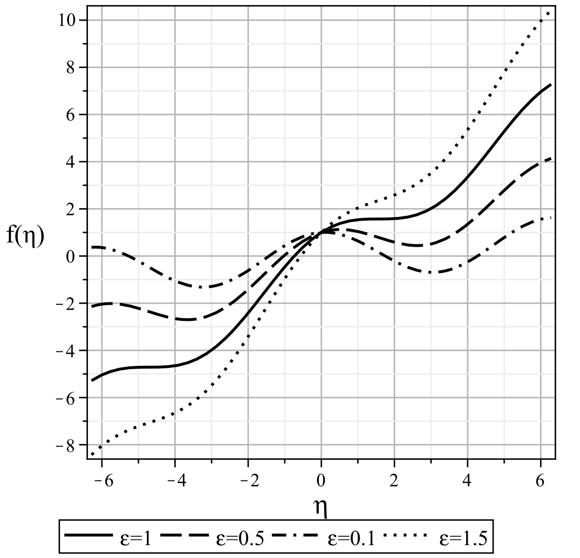


FIG. 3: Quasi-potential in dynamic variable η plotted for $\tilde{\phi} = 1$ and several values of ϵ, cf. Eq. 14.

diverges represent an inefficiency in the process. These "infinitely resonant" particles are connected to the bulk distribution by the continuous tendrils of phase space observed in our numerical simulations, cf. Fig. 2. The particles in these tendrils lessen the efficiency of the adiabatic impulse, leaving behind a continuous, topologically closed distribution with many tendrils extending away from the adiabatically displaced bulk as far in velocity space as the drive wave is accelerated.

Is there any limit in which the time-asymptotic effect of the chirped wave on the distribution is effectively indistinguishable from the effect of a DC field in a time-asymptotic sense? In other words, can a waveform with a vanishing time-averaged field mimic the cumulative effect of a DC field on an arbitrary particle distribution? To answer this question we look at the τ-integral for an arbitrary particle accelerated by the chirped wave.

The time-asymptotic τ-integral for a particle in the quasi-potential $f(\eta)$, assuming $\epsilon > 0$, is given by

$$\begin{aligned}\tau_\infty &= 2\left(\int_{-\infty}^{\eta_t} \frac{d\eta}{\dot{\eta}} - \frac{|\dot{\eta}_{-\infty}|}{ka}\right) \\ &= 2^{1/2}\int_{-\infty}^{\eta_t} \frac{d\eta}{\left[E - f(\eta)\right]^{1/2}} \\ &\quad - \frac{2\,|\dot{\eta}_{-\infty}|}{ka}\end{aligned} \tag{15}$$

with

$$\dot{\eta} = \left[2\left(E - f(\eta)\right)\right]^{1/2}, \tag{16}$$

$f(\eta)$ being the quasi-potential given in Eq. 14. Note that η_t signifies the phase of the classical turning point of the

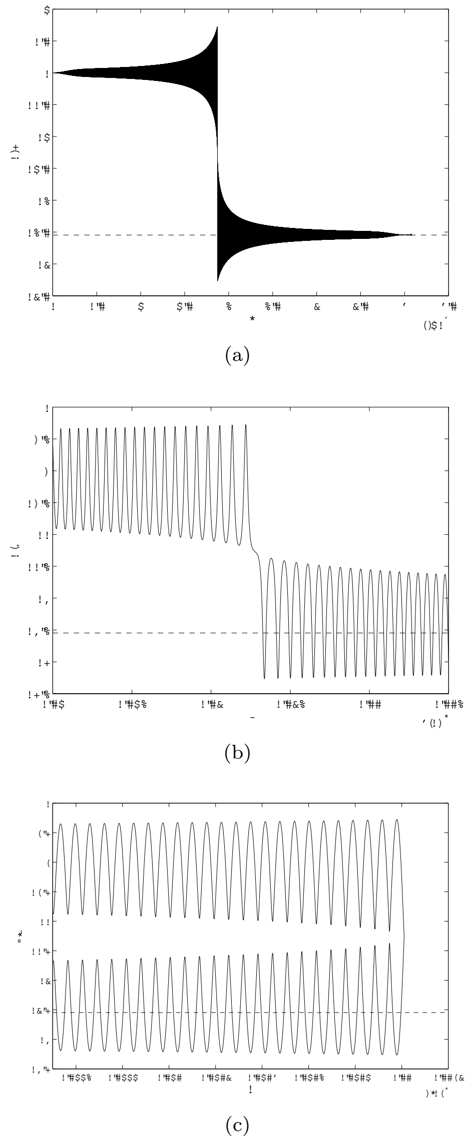


FIG. 4: (a) Net impulse delivered to initially stationary particle at the origin in the laboratory reference frame vs. time. (b) Close-up of impulse vs. time near the resonance. (c) Close-up of impulse vs. phase variable η, showing largest net impulse delivered during the fluctuation leading to exact wave-particle resonance. The dotted line shows predicted $\Delta\dot{x}$ according to time-asymptotic conservation of action, cf. Eq. 11.

particle in the quasi-potential, which itself depends on the relationship between the particle quasi-energy E, $\tilde{\phi}$, and ϵ. This form of τ_∞ is justified by the fact that if we assume the particle starts and ends infinitely far from the wave peak with which it will resonantly interact, then we observe from Eq. 4 that the "uphill" and "downhill" components of τ_∞ will be symmetric and of the same sign.

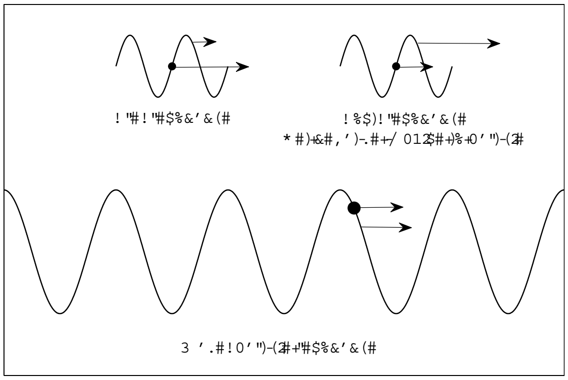


FIG. 5: Graphic illustrating the different stages of the wave-particle interaction: pre-resonance, resonance, and post-resonance.

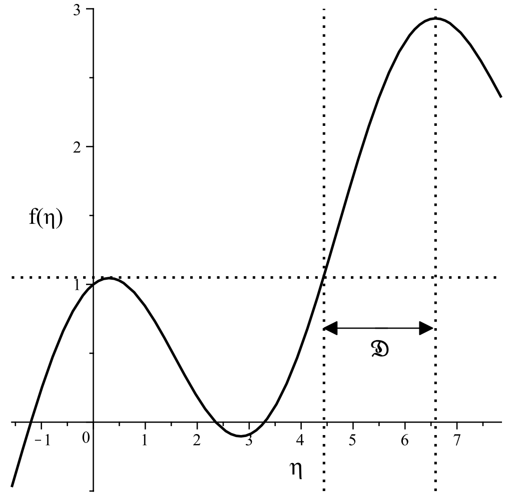


FIG. 6: Example interval of domain of validity $\mathfrak{D}_1$ for Eq. 18, corresponding to m=1 in Eq. 19

The integral in Eq. 15 corresponds to the amount of time it takes the particle to move from minus infinity to the turning point and back again, and $|\dot{\eta}_{-\infty}|$ corresponds to the particle speed at minus infinity, all in the noninertial frame of reference moving with the wave. This is an inconvenient and ambiguous form for τ_∞, because we are forced to find the difference between two infinite quanities to get a finite result. However, numerical simulations of particles interacting with an accelerating potential have shown that the majority of the net impulse delivered to the particle occurs when the particle is less than roughly half a wavelength from the point of exact wave-particle resonance; c.f. Fig. 4 for example. The particle is in exact resonance with the wave when it is traveling at the same instantaneous velocity as the wave, which corresponds to when the particle is at the classical turning point in the quasi-potential picture. The reason each η-value in Fig. 4(c) has two values for ΔV is illustrated in Fig. 5. Essentially, the particle starts off traveling faster in one direction than the wave, but as the wave accelerates, its phase velocity catches up to that of the particle, until finally the wave and particle are traveling at the same velocity. After passing through wave-particle resonance, the wave phase velocity continues to accelerate past the particle's velocity, meaning wave crests that the particle passed prior to resonance catch back up to the particle after resonance. Thus the particle interacts with each wave crest twice, and we observe two values for ΔV during the interaction: one pre-resonance and one post-resonance.

We now postulate that the salient features of the time-asymptotic wave-particle interaction will be demonstrated in the peak resonant impulse delivered to the particle during the strongest part of the resonant interaction, i.e. the largest fluctuation depicted in Fig. 4(c). The most significant variations in the outcomes of different resonant interactions should be determined by the particle dynamics very near the turning point in the quasi-potential during this peak fluctuation. On the other hand, the remainder of the interaction far from the turning point should not be strongly sensitive to the exact energy of the particle. The corrections due to the interactions far from resonance are comprised of fast oscillations that time-average to a quantity much less than the peak resonant impulse, cf. Fig. 4. The peak resonant impulse can be calculated with the following τ-integral:

$$\tau_p = 2\left[\int_{\eta_{min}}^{\eta_t} \frac{d\eta}{\dot{\eta}} - \frac{\dot{\eta}\left(\eta_{min}\right)}{ka}\right], \tag{17}$$

where η_{min} is the local minimum of $f(\eta)$ nearest the turning point, c.f. Fig. 6. Specifically, looking ahead to Eq. 19, for interval of validity $\mathfrak{D}_m$, we have $\eta_{min} = 2\pi m - \sin^{-1}\epsilon - \pi$. In order that a second order expansion of the cosine term in $f(\eta)$ will provide sufficient accuracy over the whole integration range, we split up the integration into two separate parts:

$$\tau_p = 2\left[\left(\int_{\eta_{min}}^{\eta_*} + \int_{\eta_*}^{\eta_t}\right)\frac{d\eta}{\dot{\eta}} - \frac{\dot{\eta}(\eta_{min})}{\tilde{\phi}\epsilon}\right],$$

where we note $\eta_* = (\eta_t + \eta_{min})/2$, and $ka = \tilde{\phi}\epsilon$. We now can expand the cosine term in $f(\eta)$ to second order in η about η_{min} for the first integral and about η_t for the second integral. We can define new integration variables $x \equiv \eta_t - \eta$ and $y \equiv \eta - \eta_{min}$, at which point we find

$$\tau_p \approx 2\left[\int_0^{\Theta(\eta_t)} \frac{dy}{\left[2\left(f(\eta_t) - f(\eta_{min})\right) - f''(\eta_{min})y^2\right]^{1/2}} + \int_0^{\Theta(\eta_t)} \frac{dx}{\left[2f'(\eta_t)x - f''(\eta_t)x^2\right]^{1/2}} - \frac{\dot{\eta}\left(\eta_{min}\right)}{\tilde{\phi}\epsilon}\right],$$

where $\Theta(\eta_t) \equiv (\eta_t - \eta_{min})/2$, and we used $E = f(\eta_t)$. The equation for τ_p above is general for arbitrary quasi-potential $f(\eta)$ and possesses an exact analytical solution that yields τ_p as a function of the classical turning point η_t. An expression for τ_p in the case of the sinusoidal accelerating potential in Eq. 14 is given by:

$$\tau_p(\eta_t) \approx \frac{2}{\tilde{\phi}^{1/2}} \left\{ \frac{\ln\left[1 + \Theta G + \left(\Theta^2 G^2 + 2\Theta G\right)^{1/2}\right]}{\left(\cos\eta_t\right)^{1/2}} + \frac{1}{\gamma^{1/2}} \tan^{-1}\left[\frac{(\gamma/2)^{1/2}\,\Theta}{\left[M - \gamma/2\right]^{1/2}}\right] - \frac{(2M)^{1/2}}{\epsilon} \right\}, \tag{18}$$

with

$$\begin{aligned} \gamma &= \left(1 - \epsilon^2\right)^{1/2}, \\ G(\eta_t) &= \frac{\cos\eta_t}{\epsilon - \sin\eta_t}, \\ M(\eta_t) &= \cos\eta_t + \epsilon\eta_t + \gamma + \epsilon\left(\pi + \sin^{-1}\epsilon\right). \end{aligned}$$

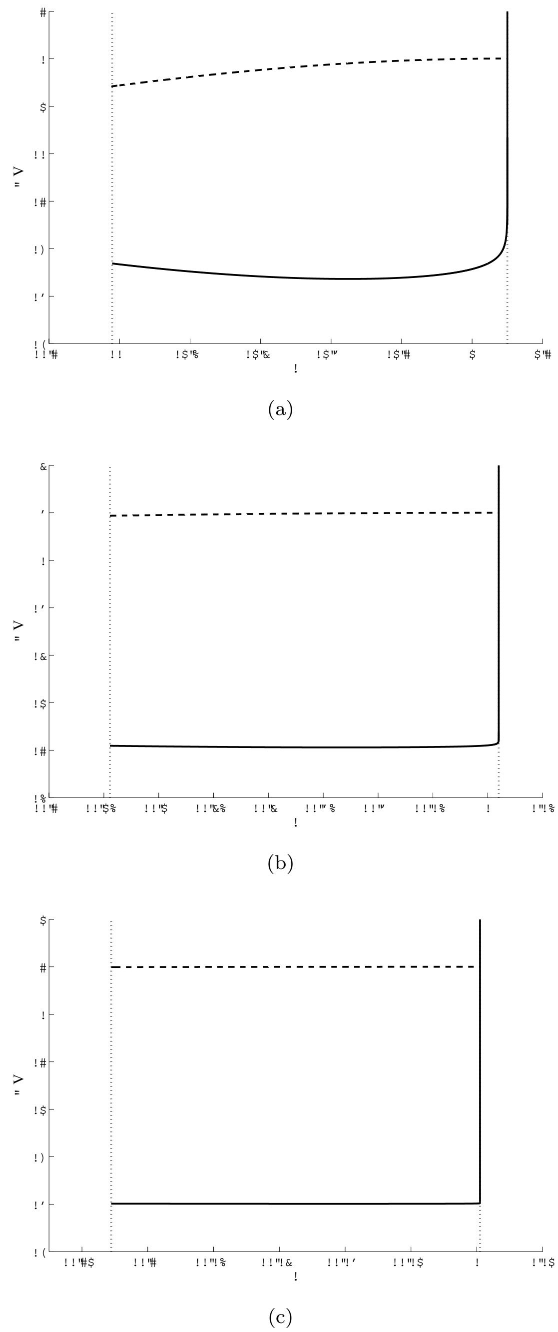
Thus we find that even an approximate representation of τ_p is a complicated function of η_t, but its scaling behavior with varying ϵ will turn out to be exactly what we expected as it transitions to the adiabatic regime. Using Eq. 5, we can now determine the approximate behavior of the peak resonant impulse delivered to the particle by the wave as a function of the particle's turning point in the noninertial reference frame moving with the accelerating wave.

The domain of validity $\mathfrak{D}$ of this expression for τ_p is composed of a countably infinite number of discontinuous intervals in η, due to the fact that a particle must energetically climb higher up a wave crest in the quasi-potential than the local maximum of the adjacent, lower wave crest in order for that point to represent a valid turning point; otherwise, the particle would have turned on the previous wave crest instead. Fig. 6 illustrates this concept. An arbitrary interval of the domain of validity is calculated to be approximately:

$$\mathfrak{D}_m = \left(2\pi m + \left[\epsilon - \left[\epsilon^2 + 2\left(1 - \sqrt{1-\epsilon^2} + \epsilon\left(2\pi - \arcsin\epsilon\right)\right)\right]^{1/2}\right], \arcsin\epsilon + 2\pi m\right], \tag{19}$$

with m any integer.

The expression in Eq. 18 demonstrates many characteristics we would expect from the full expression for τ_∞. For instance, τ_p diverges at the points of unstable equilibrium of the quasi-potential, given by $\eta_{crit} = \arcsin\epsilon \pm 2\pi n$, with n any integer; note that Eq. 18 was derived for the interval $\mathfrak{D}_0$, so here $\eta_{crit} = \arcsin\epsilon$. These points correspond to the tips of the phase space tendrils that are carried off with the wave separatrices after the wave passes through resonance with the distribution. Fig. 7 illustrates the behavior of $\Delta\dot{x}_p(\eta_t) = a\tau_p(\eta_t)$



(a) (b) (c)

FIG. 7: Analytic approximation for $\Delta\dot{x}(\eta) = a\tau_{1/4}(\eta)$ (solid line, cf. Eqs. 5 and 18) plotted against the quasi-potential $f(\eta)$ (dashed line, cf. Eq. 14) over the domain corresponding to $m = 0$ in Eq. 19 for (a) $\epsilon = 10^{-1}$, (b) $\epsilon = 10^{-2}$, and (c) $\epsilon = 10^{-3}$. Limits η_{min} and η_{max} are traced out by dotted lines.

over 3 orders of magnitude in ϵ. The right limit of the domain marks the location of η_{crit}, and note that while the singularity in the impulse is pronounced for $\epsilon = 10^{-1}$, its presence is limited to an almost trivial fraction of the domain when $\epsilon = 10^{-3}$, so small that it would not plot and had to be represented graphically using manual input of a vertical line at η_{crit}. The "bulk" behavior, on the other hand, makes a smooth transition to a nearly flat, adiabatic response as ϵ is decreased. It is almost remarkable to observe in Fig. 7(c) that $\Delta\dot{x}$ is driven from the finite, negative adiabatic response to a positive, infinite response by a nearly imperceptible shift in the value of $f(\eta)$ at η_{crit}. Thus our solution for τ_p, and hence the peak resonant impulse, appears to illustrate exactly what we had hoped to observe: a smooth transition from the non-adiabatic regime to the adiabatic regime over a few orders of magnitude of the parameter ϵ. Incidentally, one can observe that the adiabatic peak resonant impulse predicted by the solution in Fig. 7(c) closely matches the peak fluctuation encountered in the numerical simulation in Fig. 4, with a value $\Delta\dot{x}_p \approx -4$.

V. THE AC-DC CORRESPONDENCE

We now seek to confirm that the impulse delivered to particles throughout phase space scales like the adiabatic prediction in Eq. 11 as the acceleration approaches zero despite the presence of the periodic infinite divergences in $\Delta\dot{x} = a\tau$. Based on our results calculated above, we will assume that our conclusions will be equally valid if we work with the peak resonant impulse calculated in Eq. 18 rather than the full form of $a\tau_\infty$, since all of the interesting wave-particle behavior near resonance is captured in $a\tau_p$.

First, we can use Eqs. 5, 13, and 18 to solve for $\Delta\dot{x}_p$. We have:

$$\Delta\dot{x}_p = a\tau_p = \frac{\epsilon\tilde{\phi}}{k}\tau_p. \tag{20}$$

We are interested in the limit where the wave acceleration approaches zero, i.e. $\epsilon \to 0$. Referring to Eq. 19, we can show that in the regime $\epsilon \ll 1$, the interval $\mathfrak{D}_0$ goes like

$$\mathfrak{D}_0 \approx \left(\epsilon - (4\pi\epsilon)^{1/2}, \epsilon\right], \tag{21}$$

which is the interval over which our approximation for τ_p was calculated. Since Eq. 21 states that over the valid domain for Eq. 20, the value of $\eta_t \ll 1$, we can expand Eq. 18 for small ϵ *and* small η_t. Keeping corrections that are only order one in the small parameters or greater, we find

$$\Delta\dot{x}_p \approx \frac{2\epsilon\tilde{\phi}^{1/2}}{k}\left[\ln\left(\frac{\pi}{\epsilon-\eta_t}\right) + \tan^{-1}\frac{\pi}{2\sqrt{3}} - \frac{2}{\epsilon} - \frac{\pi}{2}\right]. \tag{22}$$

Taking the limit as $\epsilon \to 0$, we find that only one term survives:

$$\lim_{\epsilon\to 0}\Delta\dot{x}_p = -\frac{4\tilde{\phi}^{1/2}}{k} = -4\left(\frac{\Phi_0}{m}\right)^{1/2}, \tag{23}$$

where we used Eq. 13 to get to the final representation. Now it is apparent why we observed $\Delta\dot{x}_p \approx -4$ for both the numerical and analyical plots at the end of the last section, since both examples used parameter choices $\Phi_0 = 1 = m$.

More importantly, the scaling behavior derived in Eq. 23 is identical to the scaling behavior predicted by the adiabatic invariant model in Eq. 11; namely, the impulse goes like the square root of the potential strength divided by the mass and is independent of the scale length k of the drive wave. This correspondence between the time-asymptotic, adiabatic prediction for the net impulse and the exact peak resonant net impulse confirms our prediction that the important dynamics take place very close to resonance, and the rest of the time-asymptotic interaction is roughly the same for every particle regardless of its specific initial conditions. In fact, comparing the peak resonant impulse to the adiabatic impulse reveals the scale of the fluctuations near resonance compared to the net time-asymptotic impulse, which in the case of a sinusoidal drive wave yields a fluctuation amplitude $(4/2)/(8/\pi) \approx 0.79$ of the amplitude of the net time-asymptotic impulse.

Consider now the validity of the derivation of Eq. 23 for the case when η_t is very near ϵ; in fact, when $\eta_t = \epsilon$ the impulse is infinite, which we determined was the result of the particle coming to rest on the point of unstable equilibrium in the quasi-potential. To see that the derivation remains valid, consider the case $\eta_t = \epsilon - \delta$ and $\delta = \epsilon^\alpha$, so that the following holds:

$$\epsilon\ln\left(\frac{1}{\epsilon-\eta_t}\right) = \epsilon\ln\left(\frac{1}{\delta}\right) = \alpha\epsilon\ln\left(\frac{1}{\epsilon}\right).$$

For any finite value of α the above expression goes to zero as $\epsilon \to 0$, which means no matter how close you come to the singularity at $\eta_t = \epsilon$, the impulse will always be the adiabatic result shown in Eq. 23. This means that the density, momentum, and energy content of the tendrils of the distribution goes to zero as the wave acceleration goes to zero, since a particle would have to have an exact initial energy corresponding to $\eta_t = 2\pi m + \arcsin\epsilon$ to get caught in the tendril, and any infinitesimal fluctuation away from that energy would immediately displace it to the adiabatically-shifted portion of the distribution.

Thus we have shown that as a classical wave is adiabatically accelerated through resonance with a noninteracting Hamiltonian distribution of particles at a rate that becomes infinitely slow, its effect on the distribution becomes indistinguishable from the effect of a uniform DC field applied for a finite amount of time; namely, the impulse delivered to each particle is independent of the particle's initial conditions in phase space.

In addition, the τ-integral formalism developed to analyze the specific scenario of an accelerating sinusoidal potential can be used for a broader scope of problems, providing a systematic, tractable way to characterize the smooth transition toward the adiabatic limit of any accelerating potential waveform and the nonideal behavior associated with non-adiabatic wave-particle resonance. In practice, the τ-integral can be numerically integrated or approximated analytically, and an analytical expansion method similar to what we did above can be used to deal with the singularity near wave-particle resonance.

It is worth mentioning that simulations were carried out investigating other periodic potential configurations, such as an infinite train of widely-spaced Gaussian wave packets, and the resulting behavior of the bulk distribution is unchanged qualitatively as long as the periodic structure is accelerated adiabatically. Referring back to the original concept of current drive in a plasma suggested by Friedland *et al* [2], this general result suggests that a viable current drive scheme does not necessarily require an infinite sinusoidal beat wave or optical lattice; rather, an identical effect could be achieved using a series of pulsed, accelerating wave packets, each with some well-localized ponderomotive interaction range much smaller than the spacing between adjacent wave packet centroids.

VI. WAVE MEANS OF ACCELERATION

The concept of using waves to manipulate a particle distribution in phase space, and in particular accelerating part or all of the distribution in velocity space, is important to a number of applications. There are of course many specific wave mechanisms that have been advanced, such as recently: ponderomotive accelerations where particles exit a ponderomotive potential in an appropriate way [11–23]; ratchet effects using a cyclotron resonance [24–26]; and wakefield acceleration effects [27] including acceleration in plasma channels [28].

While there is some differentiation between the numerous aforementioned acceleration mechanisms, one fundamental premise underlies all of them; namely, there is some unavoidable sensitivity to a particle's initial conditions for the scheme to work properly, either because the particle must be injected at the right time relative to the wave in order for the interaction to be optimized, or because the particle must be ejected from the wave at a particular time in order to retain any net gain in energy. These acceleration scenarios are thus constrained in terms of overall particle throughput by the limitations of the injection and ejection methods operating in tandem with the acceleration mechanism, likely only having an optimized interaction with a small part of a broad particle phase space distribution. An acceleration scheme that makes use of the adiabatically accelerated waveform mechanism described in this paper would be largely insensitive to each particle's initial conditions and would thus operate on a much broader portion of a particle distribution with the potential of yielding much more intense and sustained particle currents. Furthermore, a distribution that is initially cold would remain at the same temperature after acceleration, meaning that high-quality monoenergetic beams could be produced without heating the particles during acceleration.

The AC-DC effect could be applied to tenuous charged particle populations immediately, where the self-fields of the distribution are negligible compared to the driving fields. However, Friedland's work with autoresonant driven plasma configurations suggests that a similar effect could be produced in dense plasmas as well [1, 2], since the plasma could be driven in such a way so that the self-field response of the plasma is phase-locked with the drive wave and preserves the coherent structure of the adiabatically accelerated waveform.

Thus, the AC-DC effect could also be used to drive electron currents in confined, quasineutral plasmas, where the accelerating wave would not only give particles traveling in the desired direction a uniform kick in velocity, but it would also take energy away from particles traveling in the other direction. As we explore in the next section, this suggests that this process could be highly efficient if the energy given back to the wave could be recirculated effectively.

VII. DISCUSSION: DC AND AC FIELDS

What we demonstrated in this paper is the limit in which wave fields could act like DC electric fields in accelerating a distribution of particles, presuming we coarse-grain time so that one full cycle of the waveform acceleration through resonance with the distribution represents one "unit" of time. The key property of a DC electric field is that the DC field accelerates all particles with uniform force regardless of the particle velocity. Because of this key property, the DC electric field can extract energy from particles that it slows down. This makes the DC field a very efficient generator of electric current; although it takes field energy to accelerate electrons to higher velocity, the energy flow is the opposite for electrons that are decelerated. In contrast, in wave-based acceleration schemes for producing current, which are based on wave-particle diffusion, the efficiency of current drive tends to be less to the extent that current is only produced as waves diffuse particles to higher energy [3]. If waves can only diffuse particles to higher energy, the opportunity to drive current by losing energy to the fields is absent, thereby diminishing the current drive efficiency.

There is an exception to this rule, but only a minor one: waves can diffuse particles to lower energy, but only when a population inversion exists along the diffusion path, such as in the presence of a density gradient [29]. However, such a situation requires not only density gradients, but also wave-particle diffusion paths that can exploit that gradient. Therefore, in general, the current drive efficiency by waves tends to be smaller than by a

DC field.

However, in terms of efficiency, making up in part for the inability of waves to produce current by extracting energy from counter-current going electrons is the ability of waves to target through wave-particle resonance conditions specific populations of charged particles. For example, the ability to target specific populations allows waves to interact selectively with electrons or with one species of ions of a specific energy. Thus, the lower-hybrid current drive (LHCD) efficiencies [30] or the electron cyclotron current drive (ECCD) efficiencies [31] can be quite high by selective acceleration of low-collisionality electrons, namely superthermal electrons. Nonetheless, while RF current drive schemes can be optimized by means of resonance conditions to have high efficiency, they still produce efficiencies not quite as high, in general, as for the Ohmic current drive obtained by a DC electric field.

It is worth mentioning that the indiscriminate nature of the DC-like acceleration exhibited by the mechanism described in this paper is *not* indiscriminate between distributions corresponding to different species. Referring to Eq. 13, we note that $\epsilon \sim m$, and so while ϵ might be very small for electrons, ions encountering the same waveform might see a much larger ϵ. Though the analysis above focused mainly on the case of small ϵ, we will note that numerical simulations revealed that as ϵ becomes of order one, the time-asymptotic impulse becomes highly sensitive to initial conditions, and of course as ϵ gets very large, the impulse delivered to a particle becomes negligible. Thus, while the DC-like acceleration mechanism might not be able to target specific populations of a single particle distribution, it *is* possible to discriminate between distributions corresponding to different particle species, making it possible, for example, to accelerate electrons efficiently while leaving the ion populations roughly unchanged.

On the other hand, apart from efficiency, generating current through RF fields has certain technological advanatages over generating current by DC electric fields. Most importantly, a DC electric field driving a toroidal current has a nonvanishing curl, so there is necessarily a monotonically time-varying magnetic field, which means that the current cannot be sustained in a purely steady state. This restriction does not apply to RF waves, which can generate purely steady state currents, thus enabling, for example, purely steady state tokamak reactors producing nuclear fusion. Waves can also be brought to bear on a plasma through different technological means than can DC fields, with a compact or remote apparatus, which might also offer in some instances a technological advantage.

Thus, the possibility of using waves, which are not restricted to pulsed operation and which utilize different power technology, to mimic the effects of DC fields could be, in principle, of considerable practical interest. There might be an opportunity to capitalize both on the high efficiency of DC-driven currents and the advantages of wave generation technology. Apart from the practical implications, it is also of significant academic interest whether waves can in fact mimic DC effects.

What we show here is that waves can in fact induce populations of charged particles to behave like they do in DC fields, but only in certain limiting cases of slowly accelerating coherent wave structures. Moreover, the phase space conservation is very different on a fine scale, and only in an averaged sense do the two effects coincide. We show that only in the limit of slowly accelerating waves does the portion of phase space not obeying DC-like behavior becomes vanishingly small.

What we do not address in depth in this paper is how these accelerating wave potentials are generated, although one possibility might be through counter-propagating wave packets that nonlinearly produce the potentials. We find it curious that an exchange of wave packets mimics in this limit the effect of a DC field, but whether there is a deeper meaning to this curiosity is left to a future study.

VIII. CONCLUSIONS

We presented the adiabatic invariant model for an adiabatically accelerating sinusoidal potential acting on a waterbag distribution, which predicts that the resonant interaction with the distribution essentially displaces the boundaries of a distribution an equal amount in velocity space in order to accommodate the empty volume moved through the distribution by the wave separatrices. Then a more rigorous analysis of the resonant impulse delivered to particles throughout phase space was carried out using the τ-integral formalism. This analysis revealed that parts of the distribution are highly resonant with the wave and deviate substantially from the prediction established by the adiabatic jump condition. Finally, we proved that if the wave phase velocity is accelerated infinitely slowly, the resonant effect of the wave on the whole distribution agrees with the adiabatic prediction and is, in fact, indistinguishable from the effect of a uniform DC field applied for a finite period of time despite the diminishing but ineffaceable presence of the highly resonant particles as the wave acceleration is decreased.

The utility of this method of particle acceleration was discussed in the context of other wave-based methods for current drive and particle acceleration. The adiabatically-accelerated wave method was noted to have the potential to exhibit near DC-like efficiency, since the wave both adds energy to particles going in the desired direction and removes energy from particles traveling in the other direction, exactly as a momentarily applied DC field would do.

IX. ACKNOWLEDGMENTS

The authors would like to thank Lazar Friedland, Pavel Khain, Jonathan Wurtele, and Ilya Dodin for useful discussions. This work was supported by US DoE contract No. DE-AC02-76-CH03073. One of us (PFS) was supported by the National Defense Science and Engineering Graduate Fellowship.

[1] L. Friedland, P. Khain, and A. G. Shagalov, Phys. Rev. Lett. **96**, 225001 (2006).
[2] P. Khain and L. Friedland, Phys. Plasmas **14**, 082110 (2007).
[3] N. J. Fisch, Rev. Mod. Phys. **59**, 175 (1987).
[4] A. M. Sessler and K. R. Symon, Proc. CERN Symposium on High Energy Accel., Geneva, p. 44 (1956).
[5] A. M. Sessler, Int. Symp. on 30 Years of Neutral Currents: From Weak Neutral Currents to the (W)/Z and Beyond, LBL-33664 (1993).
[6] A. Hofmann, Design and utilization of the SSC summer study, SLAC-PUB-3427 (1984).
[7] N. M. Kroll, P. L. Morton, and M. N. Rosenbluth, J. Quant. Elec. **17**, 8 (1981).
[8] F. Peinetti et al, Phys. Plasmas **12**, 062112 (2005).
[9] L. Friedland et al, Phys. Plasmas **11**, 4305 (2004).
[10] W. Bertsche, J. Fajans, and L. Friedland, Phys. Rev. Lett. **91**, 265003 (2003).
[11] M. N. Shneider, P. F. Barker, and S. F. Gimelshein, Appl. Phys. A **89**, 337 (2007).
[12] S. X. Hu and A. F. Starace, Phys. Rev. Lett. **88**, 245003 (2002).
[13] I. Y. Dodin and N. J. Fisch, Phys. Rev. E **68**, 056402 (2003).
[14] K. P. Singh and H. K. Malik, App. Phys. Lett. **93**, 044101 (2008).
[15] V. I. Geyko, G. M. Fraiman, I. Y. Dodin, and N. J. Fisch, arXiv:0902.1460 (2009).
[16] D. L. Bruhwiler and J. R. Cary, Phys. Rev. Lett. **68**, 255 (1992).
[17] D. L. Bruhwiler and J. R. Cary, Phys. Rev. E **50**, 3949 (1994).
[18] L. Feng and Y. Ho, Phys. Rev. E **47**, 4 (1993).
[19] L. Cicchitelli and H. Hora, IEEE J. Quantum Electronics **26**, 10 (1990).
[20] J. X. Wang et al, Phys. Rev. E **60**, 6 (1999).
[21] G. Shvets, N. J. Fisch, and A. Pukhov, IEEE Transactions on Plasma Science **28**, 4 (2000).
[22] R. R. Lindberg et al, Phys. Rev. Lett. **93**, 055001 (2004).
[23] D. N. Gupta and H. Suk, Phys. Plasmas **13**, 013105 (2006).
[24] I. Y. Dodin and N. J. Fisch, Phys. Rev. E **72**, 046602 (2005).
[25] I. Y. Dodin, N. J. Fisch, and J. M. Rax, Phys. Plasmas **11**, 5046 (2004).
[26] H. Motz and C. J. H. Watson, Adv. Electron. **23**, 153 (1967).
[27] E. Esarey, P. Sprangle, J. Krall, and A. Ting, IEEE Trans. Plasma Sci. **24**, 252 (1996).
[28] I. Y. Dodin and N. J. Fisch, Phys. Plasmas **15**, 103105 (2008).
[29] N. J. Fisch and J. M. Rax, Phys. Rev. Lett. **69**, 612 (1992).
[30] N. J. Fisch, Phys. Rev. Lett. **41**, 873 (1978);
[31] N. J. Fisch and A. H. Boozer, Phys. Rev. Lett. **45** 720 (1980).

The Princeton Plasma Physics Laboratory is operated
by Princeton University under contract
with the U.S. Department of Energy.

Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2750
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov