Princeton Plasma Physics Laboratory

PPPL- 4769

PPPL- 4769

Axiomatic Geometrical Optics, Abraham-Minkowski Controversy and Photon Properties Derived Classically

L.Y. Dodin and N.J. Fisch

June, 2012

PRINCETON PLASMA PHYSICS LABORATORY

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

Axiomatic geometrical optics, Abraham-Minkowski controversy, and photon properties derived classically

I. Y. Dodin and N. J. Fisch
Princeton Plasma Physics Laboratory, Princeton, New Jersey 08543, USA
(Dated: May 22, 2012)

By restating geometrical optics within the field-theoretical approach, the classical concept of a photon in arbitrary dispersive medium is introduced, and photon properties are calculated unambiguously. In particular, the canonical and kinetic momenta carried by a photon, as well as the two corresponding energy-momentum tensors of a wave, are derived straightforwardly from first principles of Lagrangian mechanics. The Abraham-Minkowski controversy pertaining to the definitions of these quantities is thereby resolved for linear waves of arbitrary nature, and corrections to the traditional formulas for the photon kinetic quantities are found. An application of axiomatic geometrical optics to electromagnetic waves is also presented as an example.

PACS numbers: 03.50.De, 42.50.Wk, 42.15.-i, 03.50.-z

I. INTRODUCTION

A. Motivation

The discussion about how to define the momentum and the angular momentum of a photon in dispersive medium (PDM), and even simply of a classical wave, has been revived in literature periodically during the last hundred years. The recent burst of theoretical [1–52] and experimental [53–55] publications indicates a lingering interest in the problem and, apparently, a lack of consensus or certainty about what the correct answer is. The traditional arguments can be found in reviews like Refs. [56–62] and references therein, too numerous to be listed in this paper. Let us mention only briefly that two alternative forms of the PDM momentum are adopted most commonly,

$$p_{\mathrm{M}} = \hbar\omega n_p/c, \quad p_{\mathrm{A}} = \hbar\omega/(n_g c), \tag{1}$$

known, respectively, as the Minkowski interpretation and the Abraham interpretation. (Here ω is the frequency, c is the speed of light, and $n_p = c/v_p$ and $n_g = c/v_g$ are the refraction indexes associated with, correspondingly, the phase velocity v_p and the group velocity v_g; for the two associated angular momenta see Ref. [45].) Since both have supporting theoretical and experimental evidence [1], the question about which of the two interpretations is "more correct" has been controversial.

A resolution to this Abraham-Minkowski controversy (AMC) was proposed recently in Ref. [1]. It was argued there that *both* interpretations are correct; namely, p_{M} can be attributed as the canonical momentum, and p_{A} can be attributed as the kinetic momentum of a photon. Yet, strictly speaking, the argument of Ref. [1] applies only to the case of a nonrelativistic solid dielectric. The subsequent generalization in Ref. [5] is not quite complete either; for example, the latter neglects electrostriction and magnetostriction, kinetic effects, and spatial dispersion, and also attributes v_g entirely to the Poynting flux, in disagreement with a textbook theorem [Eq. (136)]. Thus, a quantitative relativistic theory is still lacking that would correct the existing understanding of PDM, and Eqs. (1) in particular, plus extend it to waves of nonelectromagnetic nature. To offer such a theory is the purpose of this paper.

B. Field-theoretical approach

First of all let us stress that photon properties cannot be calculated until PDM itself is defined unambiguously. (In particular this means that, contrary to the common presumption, the PDM properties cannot be inferred from experiment until a comprehensive theory is developed, essentially rendering experiments redundant in this matter.) Second of all, the definition should not be expected to originate from electromagnetism, because the concept of a photon, and even of v_p and v_g that enter Eqs. (1), is not embedded in Maxwell's equations *per se*. On the other hand, the photon concept is neither entirely of quantum nature [63], and mechanical properties of quantum radiation (dipole force, radiation pressure, cooling effects on atoms, etc) are consistently shown to have direct classical analogs [64–68]. It then stands to reason that an abstract classical calculation could resolve the AMC, generalizing Eqs. (1), without assuming a specific underlying physical system whatsoever.

To understand what the right framework is for such a calculation, notice that introducing a photon implies that the frequency ω and the wave vector $\mathbf{k}$ are well defined. These are exactly the validity conditions of what is commonly known as geometrical optics (GO). (The term "optics" here means only that the theory deals with sufficiently large ω and k; i.e., waves need not be electromagnetic.) Although usually defined through rays and wave equations [69–77], the most fundamental, axiomatic GO is an abstract field theory that applies to any field having a Lagrangian density of a specific form [Eq. (11); dissipative effects can also be added, Sec. IV D]. Just like Newton's laws of particle motion hold, with obvious exceptions, independently of specific forces acting on particles, the basic GO equations are then invariant to the wave nature [78], and the wave properties can be derived in general. Hence, axiomatic GO should resolve the AMC automatically and thus will be sufficient for our purposes.

C. Outline

Here we aim to apply the GO formalism toward deriving PDM general properties deductively using nothing more that first principles of classical mechanics. In doing so, we draw on the Lagrangian field theory as elaborated in plasma physics and hydrodynamics during the last fifty years [79–107]. Since this literature, sadly, remains unknown within the mainstream approach to the AMC (with few exceptions), the general formalism of axiomatic GO will also be restated.

Specifically, below we do the following:

(a) formulate a comprehensive theory and present a tutorial on axiomatic GO, by extending and expanding on existing results;

(b) explain how the wave canonical energy-momentum tensor (EMT) is related to the photon properties in the Minkowski interpretation;

(c) introduce the wave angular momentum and spin within axiomatic GO and calculate it explicitly for cylindrical beams;

(d) derive the effect of local linear dissipation on waves and photons;

(e) unambiguously define the wave kinetic EMT and calculate it explicitly for isotropic relativistic fluids (with striction effects included);

(f) calculate the associated energy, momentum, and angular momentum per photon; show that the traditional, Abraham formulas are reproduced as a limiting case;

(g) illustrate how the properties of electromagnetic waves can be inferred deductively within axiomatic GO without appealing to Maxwell's equations for the envelope evolution.

Note that, in parts (a) and (b), which correspond to Secs. III-IV B, we mostly repeat known arguments, published previously, e.g., in Refs. [82–86]. Also keep in mind that, in application to specific media, the problem of finding both canonical and kinetic EMT of a classical electromagnetic wave was solved comprehensively in Ref. [86], which, while known within the plasma physics community, seems to remain unknown to the general readership. The difference between Ref. [86] and our paper is that we use a different machinery and arrive at results that are, in a number of aspects, more general and, as a consequence, more concise and transparent. In particular, we *refrain from specifying the wave nature*, derive photon properties, and allow for dissipation, complementing Ref. [86] on these issues.

The paper is organized as follows. In Sec. II we introduce the notation used throughout the text. In Sec. III we describe general GO waves, including nonlinear waves, in arbitrarily curved spacetime and also in the Minkowski spacetime as a particular case. In Sec. IV we reduce the theory further to describing linear waves and explain how the Minkowski representation is recovered; in particular, the wave angular momentum and dissipative effects are discussed. In Sec. V we introduce the wave kinetic EMT and reproduce the traditional formulas for the corresponding photon quantities as a limiting case. In Sec. VI we explain how the specific properties of electromagnetic waves flow deductively from the general theory. In Sec. VII we summarize our results. Auxiliary calculations are presented in Appendix, and a number of comments are also given in endnotes.

II. NOTATION

The following notation will be assumed below. We use the symbol $\doteq$ for definitions. Greek indexes span from 0 to 3 and refer to coordinates in spacetime, x^α. In particular, for the Minkowski spacetime we adopt $x^0 \doteq ct$, where c is the speed of light, and t is time. Hence the Lorentz transformation matrix, $\Lambda^\alpha{}_\beta \doteq \partial x^\alpha/\partial x'^\beta$, is given by

$$\Lambda^0{}_0 = \gamma, \quad \Lambda^0{}_i = \gamma v_i/c, \quad \Lambda^i{}_0 = \gamma v^i/c,$$
$$\Lambda^i{}_j = \delta^i_j + (\gamma - 1) v^i v_j / v^2, \tag{2}$$

where v^i is the velocity of the "primed" reference frame with respect to the laboratory frame, and $\gamma \doteq (1 - v^2/c^2)^{-1/2}$. Latin indexes i, j, and l span from 1 to 3 and refer to spatial coordinates, x^i. Spatial vectors are denoted with bold, $\mathbf{X}$; spatial tensors are also marked with hat, $\hat{\mathbf{T}}$; symbols like $\mathbf{XY} \doteq \hat{\mathbf{Z}}$ stand for spatial dyadics, $Z^{ij} = X^i Y^j$; the symbol $\hat{\mathbf{1}}$ denotes the unit spatial tensor; besides, the three-tensor

$$\hat{\mathbf{\Lambda}} \doteq \hat{\mathbf{1}} + \frac{\gamma - 1}{v^2}\,\mathbf{vv} \tag{3}$$

is the spatial part of $\Lambda^\alpha{}_\beta$. Summation over repeating indexes will also be implied; e.g., $X^i Y_i \equiv \sum_{i=1}^3 X^i Y_i$.

Latin indexes other than i, j, and l denote partial derivatives with respect to the corresponding variables (except in v_p, v_g, n_p, and n_g); e.g., for $f \doteq f(a, \omega, \mathbf{k}; t, \mathbf{x})$, the symbol $f_\mathbf{x}$ denotes the derivative (gradient) with respect to the last argument, $\mathbf{x}$. In addition to those, "full" temporal and spatial derivatives are introduced, $\partial_t \equiv \partial/\partial t$ and $\partial_i \equiv \partial/\partial x^i$, which treat *all* arguments of the function that is being differentiated as functions of t and x^i, correspondingly; e.g.,

$$\partial_t f \doteq f_a\, \partial_t a + f_\omega\, \partial_t \omega + f_{k_i} \partial_t k_i + f_t, \tag{4}$$
$$\partial_i f \doteq f_a\, \partial_i a + f_\omega\, \partial_i \omega + f_{k_j} \partial_i k_j + f_{x_i}. \tag{5}$$

The symbol ∇ denotes the associated full covariant derivative; e.g., $\nabla_i f = \partial_i f$ is the full gradient of the scalar f, and $\nabla \cdot \mathbf{F}$ is the full divergence of the vector $\mathbf{F}$,

$$\nabla \cdot \mathbf{F} = \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial x^i} \big(\sqrt{\eta}\, F^i\big), \tag{6}$$

where $\eta \doteq \det \eta_{ij}$, and $\eta_{ij} = \eta_{ji}$ is the spatial metric. [In Cartesian coordinates, Euclidean space has $\eta_{ij} = \eta^{ij} = \text{diag}\,(1,1,1)$, so $\eta = 1$.] The symbol $_{,\alpha}$ denotes the analogous (to ∂_i) full derivative with respect to x^α, and $_{;\alpha}$ denotes the analogous full covariant derivative. For example, the four-divergence is

$$F^\alpha{}_{;\alpha} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \big(\sqrt{g}\, F^\alpha\big), \tag{7}$$

where $g \doteq -\det g_{\mu\nu}$, and $g_{\mu\nu} = g_{\nu\mu}$ is the spacetime metric. For introduction to the tensor notation and index manipulation rules in particular, see Refs. [69, 108, 109].

Some specific symbols are also summarized in Table I, and the abbreviations used in the text are as follows:

ACT – action conservation theorem,
AMC – Abraham-Minkowski controversy,
EMT – energy-momentum tensor,
GO – geometrical optics,
PDM – photon in dispersive medium,
SAM – spin angular momentum,
WMS – "wave + medium" system.

III. GENERAL WAVES

A. Covariant formulation

First, let us consider a general nondissipative wave described by some action integral $S = \int \mathfrak{L}\,\sqrt{g}\,d^4x$, where

$$\sqrt{g}\,d^4x \equiv \sqrt{g}\,dx_1 dx_2 dx_3 dx_4 \tag{8}$$

is an invariant volume element in spacetime, and the four-scalar $\mathfrak{L}$ is the Lagrangian density. Since the action of the underlying medium is not included here, no invariance requirements on $\mathfrak{L}$ are imposed. Instead, we assume that the wave structure remains fixed (albeit not necessarily sinusoidal), so the wave is fully described by some canonical phase θ, which will be understood as a scalar field $\theta(x^\nu)$, and $a = a(x^\nu)$, which is an arbitrary measure of the wave amplitude [110]. We also assume that the envelope evolves on spacetime scales large compared to those of local oscillations. On such time scales, it is only the average Lagrangian density that contributes to S, so one can adopt that $\mathfrak{L}$ does not depend on θ explicitly. Instead, $\mathfrak{L}$ must depend on the phase four-gradient,

$$k_\mu \doteq \theta_{,\mu}, \tag{9}$$

which is the generalized "wave vector" (actually, a four-covector here), obviously having zero four-curl,

$$k_{\mu;\nu} - k_{\nu;\mu} = k_{\mu,\nu} - k_{\nu,\mu} = \theta_{,\mu\nu} - \theta_{,\nu\mu} = 0. \tag{10}$$

[Equation (10) is known as the consistency relation.] Besides that, $\mathfrak{L}$ must depend on a; yet the dependence on the amplitude gradients $a_{,\nu}$ is negligible in the GO limit. Thus, allowing also for slow parametric dependence on the spacetime coordinates x^ν, we postulate

$$\mathfrak{L} = \mathfrak{L}(a, k_\mu; x^\nu), \tag{11}$$

which as well can be considered as the *definition* of the GO approximation. Hence wave equations are inferred using the least action principle, namely, as follows.

First, let us consider variation of S with respect to the wave amplitude a. Since $\delta_a S = \int \mathfrak{L}_a\, \delta a\, \sqrt{g}\, d^4x$ for any δa, the requirement $\delta_a S = 0$ leads to

$$\mathfrak{L}_a = 0. \tag{12}$$

Equation (12) can be understood as the wave dispersion relation, and it is generally nonlinear, i.e., may retain essential dependence on a (see, e.g., Refs. [104, 106]).

Second, let us consider variation of S with respect to the wave phase θ [111]. Due to Eq. (9) and the fact that $\mathfrak{L}$ does not depend on θ explicitly, for any $\delta\theta$ one has

$$\begin{aligned}\delta_\theta S &= \int \mathfrak{L}_{k_\mu}\, \delta\theta_{,\mu}\, \sqrt{g}\, d^4x \\ &= \int \big[(\sqrt{g}\,\mathfrak{L}_{k_\mu}\,\delta\theta)_{,\mu} - (\sqrt{g}\,\mathfrak{L}_{k_\mu})_{,\mu}\,\delta\theta\big]\, d^4x \\ &= -\int (\mathfrak{L}_{k_\mu})_{;\mu}\, \delta\theta\, \sqrt{g}\, d^4x,\end{aligned}$$

where we used the fact that the wave field vanishes at infinity, so $\int (\ldots)_{,\mu}\, d^4x = 0$. Thus, the requirement $\delta_\theta S = 0$ yields that the four-divergence of the action flux density $\mathcal{J}^\mu \doteq -\mathfrak{L}_{k_\mu}$ is zero [112],

$$\mathcal{J}^\mu{}_{;\mu} = 0, \tag{13}$$

which is called the action conservation theorem (ACT). Since the ACT has the form of a continuity equation, one can treat $\mathcal{G}^\mu \doteq \mathcal{J}^\mu/\hbar$ as the flux density of some fictitious quasiparticles, or "photons". (In application to specific waves, one can as well think of plasmons, polaritons, or any other elementary excitations instead.) However, remember that, within our classical description, it is only the product $\hbar\mathcal{G}^\mu$ that has an explicit physical meaning, so the actual value of $\hbar$ will be irrelevant for our purposes.

Finally, let us also introduce the wave EMT as follows. Consider the (generally asymmetric) tensor

$$\mathcal{T}_\alpha{}^\beta \doteq k_\alpha \mathcal{J}^\beta + \delta_\alpha^\beta \mathfrak{L}. \tag{14}$$

The divergence of $\mathcal{T}_\alpha{}^\beta$ equals

$$\begin{aligned}\mathcal{T}_\alpha{}^\beta{}_{;\beta} &= k_{\alpha;\beta}\mathcal{J}^\beta + k_\alpha \mathcal{J}^\beta{}_{;\beta} + \delta_\alpha^\beta(\mathfrak{L}_a\, a_{;\beta} + \mathfrak{L}_{k_\lambda} k_{\lambda;\beta} + \mathfrak{L}_{x^\beta}) \\ &= k_{\alpha;\beta}\mathcal{J}^\beta + \delta_\alpha^\beta(\mathfrak{L}_{k_\lambda} k_{\lambda;\beta} + \mathfrak{L}_{x^\beta}) \\ &= k_{\alpha;\beta}\mathcal{J}^\beta - k_{\lambda;\alpha}\mathcal{J}^\lambda + \mathfrak{L}_{x^\alpha} \\ &= k_{\alpha;\beta}\mathcal{J}^\beta - k_{\alpha;\lambda}\mathcal{J}^\lambda + \mathfrak{L}_{x^\alpha} \\ &= \mathfrak{L}_{x^\alpha},\end{aligned} \tag{15}$$

TABLE I: Summarized here is some notation adopted for wave variables ("ponder." stands for ponderomotive; the integral quantities are obtained by integrating the corresponding densities over the spatial volume $dV \equiv \sqrt{\eta}\, d^3x$). The rest of the notation is explained in Sec. II and throughout the text.

	per unit spatial volume			integral	per photon	
	canonical	kinetic	ponder.	-	canonical	kinetic
number of photons	$\mathcal{N}$	-	-	N	1	-
action	$\mathcal{I}$	-	-	I	$\hbar$	-
energy	$\mathcal{E}$	ε	$\Delta\varepsilon$	-	H	h
momentum	$\boldsymbol{\mathcal{P}}$	$\boldsymbol{\rho}$	$\Delta\boldsymbol{\rho}$	-	$\mathbf{P}$	$\mathbf{p}$
angular momentum	$\boldsymbol{\mathcal{M}}$	$\boldsymbol{\mu}$	$\Delta\boldsymbol{\mu}$	-	$\mathbf{M}$	$\mathbf{m}$
photon flux	$\boldsymbol{\mathcal{G}}$	-	-	-	-	-
action flux	$\boldsymbol{\mathcal{J}}$	-	-	-	-	-
energy flux	$\boldsymbol{\mathcal{Q}}$	$\boldsymbol{\vartheta}$	-	-	-	-
momentum flux	$\hat{\mathbf{\Pi}}$	$\hat{\boldsymbol{\pi}}$	-	-	-	-
energy-momentum tensor	$\mathcal{T}$	τ	$\Delta\tau$	-	T	-
wave Lagrangian	$\mathfrak{L}$	-	-	L	-	-

where we used Eqs. (11)-(13). This tensor is then associated with the conservation law, ${\mathcal{T}_\alpha}^\beta{}_{;\beta} = 0$, yielded when the system is translationally invariant in spacetime (i.e., when the four-force is zero, $\mathfrak{L}_{x^\alpha} = 0$). Hence, ${\mathcal{T}_\alpha}^\beta$ is a true canonical EMT [113], as one could also infer from the standard definition that is based on Noether's theorem [114]. However, notice that, in contrast with the fundamental theorem of the vacuum field theory [69, Sec. 32], $\mathcal{T}^{\alpha\beta}$ does not permit the usual [86, 99, 115–118] symmetrization, since $\mathfrak{L}$ is not restricted by any invariance requirements [79]. In particular the very fact that a scalar field such as $\theta(x^\nu)$ yields an asymmetric EMT already proves the lack of Lorentz invariance [119, Sec. 5.6].

B. Application to the Minkowski spacetime

From now on, we will assume the Minkowski spacetime with metric signature $(-,+,+,+)$; hence,

$$g_{00} = g^{00} = -1, \quad \eta_{ij} \doteq g_{ij}, \quad \eta = g. \tag{16}$$

(Although the space is Euclidean, we will allow for curvilinear coordinates; thus, albeit flat, the spatial metric η_{ij} can otherwise be arbitrary.) In this case, $k_\alpha = (-\omega/c, \mathbf{k})$, and $k^\alpha = (\omega/c, \mathbf{k})$, where

$$\omega \doteq -\partial_t \theta, \quad \mathbf{k} \doteq \nabla\theta. \tag{17}$$

Then Eq. (10) turns into the following set of equations:

$$\partial_t \mathbf{k} + \nabla\omega = 0, \quad \nabla \times \mathbf{k} = 0. \tag{18}$$

One may notice also that the latter equation here can be considered as the *initial condition* for the former one, taking curl of which readily yields $\partial_t(\nabla \times \mathbf{k}) = 0$.

Accordingly, Eq. (11) becomes

$$\mathfrak{L} = \mathfrak{L}(a, \omega, \mathbf{k}; t, \mathbf{x}). \tag{19}$$

The dispersion relation hence holds in the form (12). The ACT can be rederived from Eq. (19) or it can be deduced from Eq. (13) by substituting $\mathcal{J}^\alpha = (c\mathcal{I}, \boldsymbol{\mathcal{J}})$; either way, one gets (cf. Refs. [83, 84])

$$\partial_t \mathcal{I} + \nabla \cdot \boldsymbol{\mathcal{J}} = 0, \tag{20}$$

where $\mathcal{I}$ is the action density, and $\boldsymbol{\mathcal{J}}$ is the action spatial flux density, introduced as follows:

$$\mathcal{I} \doteq \mathfrak{L}_\omega, \quad \boldsymbol{\mathcal{J}} \doteq -\mathfrak{L}_{\mathbf{k}}. \tag{21}$$

In particular, integration of Eq. (20) over the volume $dV \equiv \sqrt{\eta}\, d^3x$ yields conservation of the integral action,

$$I \doteq \int \mathcal{I}\, dV = \text{const}. \tag{22}$$

Introducing the photon density $\mathcal{N} \doteq \mathcal{I}/\hbar$ and the photon spatial flux density $\boldsymbol{\mathcal{G}} \doteq \boldsymbol{\mathcal{J}}/\hbar$, one can further rewrite Eq. (20) as $\partial_t \mathcal{N} + \nabla \cdot \boldsymbol{\mathcal{G}} = 0$, and Eq. (22) will yield the photon conservation, $N \doteq \int \mathcal{N}\, dV = \text{const}$. Also notice that both I and N are Lorentz invariants, as well-known to flow from the general (unlike, e.g., in Ref. [120]) properties of the continuity equation [109, Sec. 2.6].

The elements of the (contravariant) EMT are now

$$\mathcal{T}^{00} = \omega\mathcal{I} - \mathfrak{L}, \quad \mathcal{T}^{0i} = \omega\mathcal{J}^i/c,$$
$$\mathcal{T}^{i0} = ck^i\mathcal{I}, \quad \mathcal{T}^{ij} = k^i\mathcal{J}^j + \eta^{ij}\,\mathfrak{L}. \tag{23}$$

In particular, Eq. (15) yields

$$\frac{\partial \mathcal{T}^{00}}{\partial t} + \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial x^i} (c\mathcal{T}^{0i}\sqrt{\eta}) = w, \tag{24}$$

which is a continuity equation for $\mathcal{T}^{00}$ with the right-hand side being $w \doteq g^{00}c\mathfrak{L}_{x^0} = -\mathfrak{L}_t$. Since the latter has the meaning of the canonical power source, $\mathcal{E} \doteq \mathcal{T}^{00}$ must be the wave canonical energy density, and $\mathcal{Q}^i \doteq c\mathcal{T}^{0i}$ must be the canonical energy flux density. Similarly,

$$\frac{1}{c}\frac{\partial \mathcal{T}^{i0}}{\partial t} + \frac{1}{\sqrt{\eta}}\frac{\partial}{\partial x^j}\,(\mathcal{T}^{ij}\sqrt{\eta}) = f^i, \tag{25}$$

which is a continuity equation for the three-vector $\mathcal{T}^{i0}/c$ with the right-hand side being $\mathbf{f} \doteq \mathfrak{L}_{\mathbf{x}}$. Since the latter has the meaning of the canonical momentum source, $\mathcal{P}^i \doteq \mathcal{T}^{i0}/c$ must be the wave canonical momentum density, and the (generally asymmetric) three-tensor $\Pi^{ij} \doteq \mathcal{T}^{ij}$ must be the canonical momentum flux density [121].

In summary, one then has

$$\mathcal{T}^{\alpha\beta} = \begin{pmatrix} \mathcal{E} & \boldsymbol{\mathcal{Q}}/c \\ c\boldsymbol{\mathcal{P}} & \hat{\boldsymbol{\Pi}} \end{pmatrix}, \tag{26}$$

where the individual blocks are given by

$$\begin{aligned} &\mathcal{E} = \omega\mathcal{I} - \mathfrak{L}, \quad \boldsymbol{\mathcal{Q}} = \omega\boldsymbol{\mathcal{J}}, \\ &\boldsymbol{\mathcal{P}} = \mathbf{k}\mathcal{I}, \quad \hat{\boldsymbol{\Pi}} = \mathbf{k}\boldsymbol{\mathcal{J}} + \mathfrak{L}\,\hat{\mathbf{1}}, \end{aligned} \tag{27}$$

and Eqs. (24) and (25) can be written as follows:

$$\partial_t\mathcal{E} + \nabla\cdot\boldsymbol{\mathcal{Q}} = w, \quad \partial_t\boldsymbol{\mathcal{P}} + \nabla\cdot\hat{\boldsymbol{\Pi}} = \mathbf{f}. \tag{28}$$

It is hence seen that the wave energy propagates at velocity $\boldsymbol{\mathcal{Q}}/\mathcal{E}$ that is generally different from the action flow velocity $\boldsymbol{\mathcal{J}}/\mathcal{I}$ [cf. Eq. (20)], and similarly for the momentum flow velocity. Moreover, those three turn out to be different from the velocities of *information*, or the nonlinear group velocities, of which there can also be more than one. For an expanded discussion on this see Refs. [84, 107] and references therein.

IV. LINEAR WAVES: MINKOWSKI REPRESENTATION

A. Basic equations

Now let us consider a linear wave, i.e., such that has $\omega(\mathbf{k};t,\mathbf{x})$ independent of a. In this case, from Eq. (12) it is seen that $\mathfrak{L}_a$ must be separable as $\mathfrak{L}_a = \mathfrak{D}(\omega,\mathbf{k})A_a$, where $A(a,\omega,\mathbf{k})$ is some function such that A_a is nonzero. [Parametric dependence of functions like $\mathfrak{L}$, $\mathfrak{D}$, and A on $(t,\mathbf{x})$ is also implied but will be omitted for the sake of brevity.] Then,

$$\mathfrak{L} = \mathfrak{D}(\omega,\mathbf{k})A. \tag{29}$$

It will hence be convenient to think of a as of a *linear* measure of the oscillating field amplitude. Then, most commonly, one will have $A \propto a^2$; yet for our purposes the actual dependence need not be specified.

Equation (12) now yields

$$\mathfrak{D}(\omega,\mathbf{k}) = 0. \tag{30}$$

Thus Eqs. (21) become

$$\mathcal{I} = \mathfrak{D}_\omega A, \quad \boldsymbol{\mathcal{J}} = -\mathfrak{D}_{\mathbf{k}}A, \tag{31}$$

and Eqs. (27) take the form

$$\mathcal{E} = \omega\mathcal{I}, \quad \boldsymbol{\mathcal{Q}} = \omega\boldsymbol{\mathcal{J}}, \quad \boldsymbol{\mathcal{P}} = \mathbf{k}\mathcal{I}, \quad \hat{\boldsymbol{\Pi}} = \mathbf{k}\boldsymbol{\mathcal{J}}. \tag{32}$$

Hence the photon canonical energy, $H \doteq \mathcal{E}/\mathcal{N}$, and the photon canonical momentum, $\mathbf{P} \doteq \boldsymbol{\mathcal{P}}/\mathcal{N}$, equal [82]

$$H = \hbar\omega, \quad \mathbf{P} = \hbar\mathbf{k}, \tag{33}$$

matching the Minkowski interpretation *exactly* and independently of the wave nature. [In fact, $\mathbf{P} = \hbar\mathbf{k}$ holds even for nonlinear waves; cf. Eqs. (27).] In particular, $P^\alpha \doteq (H/c,\mathbf{P}) = \hbar k^\alpha$ happens to be a true four-vector, by definition of k^α, so $P^\alpha P_\alpha$ is a Lorentz invariant. The latter can also be understood as a measure of the photon canonical mass $\mathfrak{M}$, defined via

$$\mathfrak{M}^2 \doteq -P^\alpha P_\alpha/c^2 \tag{34}$$

(cf., e.g., Refs. [101, 122, 123]).

Further, differentiating Eq. (30) with respect to $\mathbf{k}$ [with $\omega = \omega(\mathbf{k};t,\mathbf{x})$] also gives $\mathfrak{D}_\omega\mathbf{v}_g + \mathfrak{D}_{\mathbf{k}} = 0$, where we introduced the linear group velocity $\mathbf{v}_g \doteq \omega_{\mathbf{k}}$; therefore,

$$\mathbf{v}_g = -\mathfrak{D}_{\mathbf{k}}/\mathfrak{D}_\omega = \boldsymbol{\mathcal{J}}/\mathcal{I}. \tag{35}$$

Hence, Eq. (26) yields $\mathcal{T}^{\alpha\beta} = \mathcal{N}T^{\alpha\beta}$, where

$$T^{\alpha\beta} = \begin{pmatrix} \hbar\omega & \hbar\omega\mathbf{v}_g/c \\ c\hbar\mathbf{k} & \hbar\mathbf{k}\mathbf{v}_g \end{pmatrix} \tag{36}$$

is the canonical EMT per photon. Alternatively, one can also exclude $\mathcal{N}$ and rewrite Eqs. (32) as

$$\boldsymbol{\mathcal{P}} = \mathbf{k}\mathcal{E}/\omega, \quad \boldsymbol{\mathcal{Q}} = \mathcal{E}\mathbf{v}_g, \quad \hat{\boldsymbol{\Pi}} = \boldsymbol{\mathcal{P}}\mathbf{v}_g. \tag{37}$$

It is seen, from here and Eqs. (28), that the canonical action, energy, and momentum are all transported at the same velocity, $\mathbf{v}_g$. However, keep in mind that the full, or "kinetic" energy and momentum densities carried by the wave (Sec. V) generally do not have this property.

Finally, let us introduce photon trajectories, $d_t\mathbf{x} = \mathbf{v}_g$, also known as GO rays. Along those trajectories,

$$d_t = \partial_t + \mathbf{v}_g\cdot\nabla. \tag{38}$$

Then Eqs. (18) yield

$$d_t\mathbf{x} = \mathbf{v}_g, \quad d_t\mathbf{k} = -\omega_{\mathbf{x}}, \quad d_t\omega = \omega_t. \tag{39}$$

[Remember that the derivatives $\omega_{\mathbf{x}}$ and ω_t of $\omega(\mathbf{k};t,\mathbf{x})$ are taken at fixed $\mathbf{k}$.] In particular, the ACT can hence be written as

$$d_t\ln\mathcal{I} = -\nabla\cdot\mathbf{v}_g. \tag{40}$$

Also notice that Eqs. (39) can be understood as canonical equations for the photon motion governed by the Hamiltonian $H(\mathbf{x}, \mathbf{P}; t)$. In this form, i.e.,

$$d_t\mathbf{x} = H_{\mathbf{P}}, \quad d_t\mathbf{P} = -H_{\mathbf{x}}, \quad d_tH = H_t, \tag{41}$$

they are identical to the motion of a true classical particle such as an electron, which supports the well-known analogy between GO and classical mechanics [124, Sec. 9.8]. Reverting to Eqs. (11) and (19), it is seen then that not just waves, but classical particles too can be described in terms of phases and amplitudes [125].

B. Noether's integrals

Various transport equations can now be derived from

$$\begin{aligned}
&\partial_t(\mathsf{X}\mathcal{I}) + \nabla\cdot(\mathsf{X}\boldsymbol{\mathcal{J}}) = \\
&= (\partial_t\mathsf{X})\mathcal{I} + \mathsf{X}(\partial_t\mathcal{I}) + (\nabla\mathsf{X})\boldsymbol{\mathcal{J}} + \mathsf{X}(\nabla\cdot\boldsymbol{\mathcal{J}}) \\
&= (\partial_t\mathsf{X})\mathcal{I} + (\nabla\mathsf{X})\boldsymbol{\mathcal{J}} \\
&= \mathcal{I}\,(\partial_t\mathsf{X} + \mathbf{v}_g\cdot\nabla\mathsf{X}) \\
&= \mathcal{I}\,d_t\mathsf{X},
\end{aligned} \tag{42}$$

which holds for arbitrary X. Some of those are as follows.

Action. — Taking X equal to a constant, one recovers Eq. (20), or the ACT. [Of course, this is not an independent derivation of the ACT, since the latter itself was used in deriving Eq. (42).] As already emphasized, Eq. (20) is due to the fact that $\mathfrak{L}$ does not depend on θ explicitly. Since it also implies conservation of the integral action I, the latter can be understood as the corresponding Noether's integral.

Energy. — Taking $\mathsf{X} = \omega$, one obtains

$$\partial_t\mathcal{E} + \nabla\cdot(\mathcal{E}\mathbf{v}_g) = \mathcal{I}\,d_t\omega. \tag{43}$$

As seen from Eq. (39), in stationary medium $d_t\omega = 0$, so one recovers the result obtained in Sec. III, namely, that the wave integral energy, $\int\mathcal{E}\,dV$, is the Noether's integral that is conserved when the system is translationally invariant in time. Another corollary, which is obtained by comparing Eq. (43) with Eq. (24), is that

$$-\mathfrak{L}_t = w = \mathcal{I}\,d_t\omega = \mathcal{I}\,\omega_t, \tag{44}$$

where we also used Eq. (39). Alternatively, one can rewrite this as $w = \mathcal{N}\,d_tH$, where d_tH is the work on an individual photon per unit time.

Momentum. — Taking $\mathsf{X} = \mathbf{k}$, one obtains

$$\partial_t\boldsymbol{\mathcal{P}} + \nabla\cdot(\boldsymbol{\mathcal{P}}\mathbf{v}_g) = \mathcal{I}\,d_t\mathbf{k}. \tag{45}$$

As seen from Eq. (39), in homogeneous medium $d_t\mathbf{k} = 0$, so one recovers the result obtained in Sec. III, namely, that the wave integral momentum, $\int\boldsymbol{\mathcal{P}}\,dV$, is the Noether's integral that is conserved when the system is translationally invariant in space. Another corollary, which is obtained by comparing Eq. (45) with Eq. (25), is that

$$\mathfrak{L}_{\mathbf{x}} = \mathbf{f} = \mathcal{I}\,d_t\mathbf{k} = -\mathcal{I}\,\omega_{\mathbf{x}}, \tag{46}$$

where we also used Eq. (39). Alternatively, one can rewrite this as $\mathbf{f} = \mathcal{N}\,d_t\mathbf{P}$, where $d_t\mathbf{P}$ is the force on an individual photon.

Angular momentum. — Taking $\mathsf{X} = \mathbf{x}\times\mathbf{k}$, one obtains from Eq. (42) that

$$\partial_t\boldsymbol{\mathcal{M}} + \nabla\cdot(\boldsymbol{\mathcal{M}}\mathbf{v}_g) = \mathcal{I}\,d_t(\mathbf{x}\times\mathbf{k}), \tag{47}$$

where we formally introduced $\boldsymbol{\mathcal{M}} \doteq (\mathbf{x}\times\mathbf{k})\mathcal{I}$, or

$$\boldsymbol{\mathcal{M}} = \mathbf{x}\times\boldsymbol{\mathcal{P}}. \tag{48}$$

Based on Eq. (48), one could anticipate that $\boldsymbol{\mathcal{M}}$ is the wave angular momentum density, and indeed Eq. (47) yields that this is the case, as we will now prove.

C. Angular momentum

Conservation theorem. — Consider system rotation by an arbitrary infinitesimal angle $\delta\boldsymbol{\varphi}$. Associated with this rotation will be a variation of the Lagrangian density

$$\delta\mathfrak{L} = \mathfrak{L}_{\mathbf{k}}\cdot\delta\mathbf{k} + \mathfrak{L}_{\mathbf{x}}\cdot\delta\mathbf{x}, \tag{49}$$

where we substituted Eq. (11) for $\mathfrak{L}_a$; also,

$$\delta\mathbf{k} = \delta\boldsymbol{\varphi}\times\mathbf{k}, \quad \delta\mathbf{x} = \delta\boldsymbol{\varphi}\times\mathbf{x}, \tag{50}$$

$\mathfrak{L}_{\mathbf{k}} = -\boldsymbol{\mathcal{J}} = -\mathbf{v}_g\mathcal{I}$, and $\mathfrak{L}_{\mathbf{x}} = \mathcal{I}\,d_t\mathbf{k}$, where the latter is taken from Eq. (46). Hence,

$$\begin{aligned}
\mathcal{I}^{-1}\delta\mathfrak{L} &= -\mathbf{v}_g\cdot(\delta\boldsymbol{\varphi}\times\mathbf{k}) + d_t\mathbf{k}\cdot(\delta\boldsymbol{\varphi}\times\mathbf{x}) \\
&= \delta\boldsymbol{\varphi}\cdot(\mathbf{v}_g\times\mathbf{k}) + \delta\boldsymbol{\varphi}\cdot(\mathbf{x}\times d_t\mathbf{k}) \\
&= \delta\boldsymbol{\varphi}\cdot d_t(\mathbf{x}\times\mathbf{k}).
\end{aligned} \tag{51}$$

Having $\delta\mathfrak{L} = 0$ yields that $d_t(\mathbf{x}\times\mathbf{k}) = 0$. From Eq. (47), one then obtains that

$$\partial_t\boldsymbol{\mathcal{M}} + \nabla\cdot(\boldsymbol{\mathcal{M}}\mathbf{v}_g) = 0, \tag{52}$$

which means, in particular, that $\int\boldsymbol{\mathcal{M}}\,dV$ is conserved. Since this is the invariant associated with the medium isotropy, it by definition [126, Sec. 9] represents the wave angular momentum. Correspondingly, $\boldsymbol{\mathcal{M}}$ is the wave angular momentum density [127]. Also, $\mathbf{M} \doteq \boldsymbol{\mathcal{M}}/\mathcal{N}$, or

$$\mathbf{M} = \mathbf{x}\times\mathbf{P}, \tag{53}$$

is the angular momentum of a photon, $\hbar\,d_t(\mathbf{x}\times\mathbf{k}) \equiv d_t\mathbf{M}$ is the torque on a photon (cf. Ref. [94]), and the corresponding dynamic equation is spelled out as

$$d_t\mathbf{M} = \mathbf{v}_g\times\mathbf{P} - \mathbf{x}\times H_{\mathbf{x}}. \tag{54}$$

Spin angular momentum (SAM). — Consider a stationary wave beam symmetric with respect to z axis; i.e., in cylindrical coordinates (r,ϕ,z), the amplitude a and the wave vector components k_r, k_ϕ, k_z are independent of ϕ. The consistency relation (18) requires then that $\partial_r(rk_\phi)=0$, so $k_\phi = m/r$, where m is a constant. This gives $\mathcal{M}_z = rk_\phi\mathcal{I} = m\mathcal{I}$, or that the carried angular momentum per photon is $M_z = m\hbar$. To find m, notice that, due to $k_\phi = r^{-1}\partial_\phi\theta$, the wave canonical phase has the form $\theta = m\phi - \omega t + \Xi(r,z)$, where Ξ is some function of r and z only. Thus, after any time δt, the wave must repeat itself, at the same r and z, in the coordinate frame rotated by $\delta\phi = (\omega/m)\,\delta t$. Satisfying this condition are, in fact, only circularly polarized waves (at least, in free space), corresponding to $m=\pm 1$. Other types of wave beams therefore cannot be considered symmetric within GO and thus can be assigned only average m. Specifically, decomposing a wave with a given elliptic polarization into the two independent circularly-polarized components with corresponding weights C_+ and C_-, one gets $\langle m\rangle = C_+ - C_-$. In particular, linear polarization corresponds to $C_+ = C_-$, in which case $\langle m\rangle = 0$.

These results match the known quantum theorem, which says that states with circular polarization are the only polarization states of a free photon that are eigenstates of the corresponding SAM projection, $M_z = \pm\hbar$ [128, Sec. 8]. Thus, for an axially symmetric beam, $\mathcal{M}_z$ that originates entirely from the beam polarization can be called the SAM density. Interestingly, it can also be interpreted as follows. For those (circularly polarized) waves that do allow precise definition of the SAM, the latter appears due to the singularity of k_ϕ at $r=0$, i.e., due to $\theta(r=0)$ being undefined [129]. In this sense, the canonical phase increment $\Delta\theta = 2\pi m$ along a closed contour encircling the symmetry axis is the corresponding Berry phase [130, 131], so the photon SAM (in units $\hbar$) is nothing but the Berry index of the classical phase field.

Finally, note that a wave beam that is not axially symmetric will also carry additional, "orbital" momentum [132, 133]. The latter is included in Eq. (48), and separating it from the SAM unambiguously may not be possible except in special cases, as usual; see, e.g., Ref. [132–134] or Ref. [128, Sec. 6].

D. Dissipation

Suppose now that a linear wave experiences weak dissipation. Then, comprising the wave locally are Fourier harmonics with *complex* frequencies and wave vectors,

$$\Omega = \Omega' + i\Omega'', \quad \mathbf{K} = \mathbf{K}' + i\mathbf{K}''. \tag{55}$$

Assuming the local dispersion relation in the form

$$\mathfrak{D}(\Omega,\mathbf{K}) = 0, \tag{56}$$

let us keep only the terms of the zeroth and first order in Ω'' and $\mathbf{K}''$. Then one gets

$$\mathfrak{D} + i\mathfrak{D}_\Omega\Omega'' + i\mathfrak{D}_\mathbf{K}\cdot\mathbf{K}'' = 0, \tag{57}$$

where $\mathfrak{D}$ and its derivatives are henceforth evaluated at $(\Omega',\mathbf{K}')$. Now suppose $\mathfrak{D} = \mathfrak{D}' + i\mathfrak{D}''$, where $\mathfrak{D}'' \doteq \operatorname{Im}\mathfrak{D}$ is much smaller than $\mathfrak{D}' \doteq \operatorname{Re}\mathfrak{D}$. One hereby obtains

$$\mathfrak{D}' + i\mathfrak{D}'' + i\mathfrak{D}'_\Omega\Omega'' + i\mathfrak{D}'_\mathbf{K}\cdot\mathbf{K}'' = 0 \tag{58}$$

(where higher-order terms were neglected), the real part and the imaginary part of which are, correspondingly,

$$\mathfrak{D}' = 0, \tag{59}$$

$$\mathfrak{D}'' + \mathfrak{D}'_\Omega\Omega'' + \mathfrak{D}'_\mathbf{K}\cdot\mathbf{K}'' = 0. \tag{60}$$

From here, the envelope dynamics is inferred as follows.

At any given time, the field distribution of the real system can be mapped into the auxiliary nondissipative system, where the wave phase θ is well defined, and

$$\mathfrak{L} \doteq \mathfrak{D}'(\omega,\mathbf{k})A. \tag{61}$$

This defines the instantaneous a and also the instantaneous *real* canonical frequency and wave vector, $(\omega,\mathbf{k})$; hence all other local quantities can be introduced through $\mathfrak{L}(a,\omega,\mathbf{k})$ too. However, the dynamics in the auxiliary system and in the real system are different; thus, for the latter, an extra term Γ must be added in Eq. (40),

$$d_t \ln\mathcal{I} = -\nabla\cdot\mathbf{v}_g - \Gamma. \tag{62}$$

Assume that dissipation is determined by the local $(a,\omega,\mathbf{k})$ and by the local parameters of the medium, rather than their gradients. Then one can find Γ by calculating it for homogeneous stationary medium and a wave whose field is locally "monochromatic", i.e., can be assigned particular complex $(\Omega,\mathbf{K})$ [which map to the given canonical $(\omega,\mathbf{k})$]. Then,

$$\Gamma = -d_t\ln\mathcal{I} = -\varkappa(\Omega'' - \mathbf{v}_g\cdot\mathbf{K}''), \tag{63}$$

where $\varkappa \doteq d\ln A/d\ln a$ (which commonly equals 2; see Sec. IV A), and the left-hand side is evaluated at $(\Omega',\mathbf{K}')$. On the other hand, Eq. (60) yields

$$\Omega'' - \mathbf{v}_g\cdot\mathbf{K}'' = -\mathfrak{D}''/\mathfrak{D}'_\Omega. \tag{64}$$

Hence Γ is connected with the dispersion function as

$$\Gamma(\omega,\mathbf{k}) = \varkappa\,\mathfrak{D}''(\omega,\mathbf{k})/\mathfrak{D}'_\omega(\omega,\mathbf{k}), \tag{65}$$

where we used that, to the leading order, it is sufficient to take $(\Omega',\mathbf{K}')\approx(\omega,\mathbf{k})$ on the right-hand side.

Now let us present the corresponding transport equations. Similarly to Eq. (42), one has, for any X, that

$$\partial_t(\mathsf{X}\mathcal{I}) + \nabla\cdot(\mathsf{X}\boldsymbol{\mathcal{J}}) = \mathcal{I}\,d_t\mathsf{X} - \Gamma\mathsf{X}\mathcal{I}. \tag{66}$$

Since X is arbitrary, the number of equations that can be produced from here is infinite, like in Sec. IV B. In particular, those for the action, the energy, the momentum, and the angular momentum are obtained by taking

$\mathsf{X}=1$, $\mathsf{X}=\omega$, $\mathsf{X}=\mathbf{k}$, and $\mathsf{X}=\mathbf{x}\times\mathbf{k}$, correspondingly, and are as follows:

$$\partial_t\mathcal{I}+\nabla\cdot(\mathcal{I}\mathbf{v}_g)=-\Gamma\mathcal{I}, \tag{67}$$

$$\partial_t\mathcal{E}+\nabla\cdot(\mathcal{E}\mathbf{v}_g)=\mathcal{I}\,d_t\omega-\Gamma\mathcal{E}, \tag{68}$$

$$\partial_t\boldsymbol{\mathcal{P}}+\nabla\cdot(\boldsymbol{\mathcal{P}}\mathbf{v}_g)=\mathcal{I}\,d_t\mathbf{k}-\Gamma\boldsymbol{\mathcal{P}}, \tag{69}$$

$$\partial_t\boldsymbol{\mathcal{M}}+\nabla\cdot(\boldsymbol{\mathcal{M}}\mathbf{v}_g)=\mathcal{I}\,d_t(\mathbf{x}\times\mathbf{k})-\Gamma\boldsymbol{\mathcal{M}}. \tag{70}$$

The physical statement contained in these is twofold. First of all, one can see that the decay rate is the same in all the equations, regardless of the specific X. [This, of course, is seen already from Eq. (66).] Second of all, this rate is actually *known* from Eq. (65), which connects Γ with the dispersion function $\mathfrak{D}$. In particular, the action loss per unit volume per unit time can be written as

$$\imath_{\text{loss}}\doteq\Gamma\mathcal{I}=\varkappa\,\mathfrak{D}''a^2, \tag{71}$$

and the corresponding losses of the wave energy, momentum, and angular momentum are given by

$$w_{\text{loss}}=\omega\imath_{\text{loss}},\quad \mathbf{f}_{\text{loss}}=\mathbf{k}\imath_{\text{loss}},\quad \boldsymbol{\kappa}_{\text{loss}}=(\mathbf{x}\times\mathbf{k})\imath_{\text{loss}}.$$

Also notice that $d_t\omega$ and $d_t\mathbf{k}$ entering Eqs. (67)-(70) can be taken from the GO ray equations. Since based entirely on Eqs. (17) and (18) (Sec. IV A), those happen to be unaffected by dissipation; i.e., they are still given by Eqs. (39). Hence, the above results can be interpreted as follows: local dissipation does not affect individual photons but rather changes the photon density.

For an explanation of how the results reported here apply to electromagnetic waves, see Sec. VI. The same results are also applicable to dissipation-driven instabilities ($\Gamma<0$). Nondissipative instabilities can be accommodated within GO too, namely, by allowing for complex rays; for details see Ref. [135] and references therein.

V. LINEAR WAVES: ABRAHAM REPRESENTATION

A. Basic definitions

In addition to the wave canonical, or Minkowski EMT that we discussed so far, one can also introduce the corresponding so-called kinetic, or Abraham EMT,

$$\tau^{\alpha\beta}=\begin{pmatrix}\varepsilon & \boldsymbol{\vartheta}/c\\ c\boldsymbol{\rho} & \hat{\boldsymbol{\pi}}\end{pmatrix}. \tag{72}$$

It is defined such that, being a part of the complete EMT that describes the "wave + medium" system (WMS), $\tau^{\alpha\beta}$ comprises all the wave-related (i.e., a-dependent) dynamics of the medium and fields. We hence express it as $\tau^{\alpha\beta}=\mathcal{T}^{\alpha\beta}+\Delta\tau^{\alpha\beta}$, where $\Delta\tau^{\alpha\beta}$ is the "ponderomotive" part that is stored in the medium, and, similarly,

$$\varepsilon=\mathcal{E}+\Delta\varepsilon,\quad \boldsymbol{\rho}=\boldsymbol{\mathcal{P}}+\Delta\boldsymbol{\rho},\quad \boldsymbol{\mu}=\boldsymbol{\mathcal{M}}+\Delta\boldsymbol{\mu}. \tag{73}$$

In particular, notice the following. Since the WMS is closed and thus Lorentz-invariant, its complete EMT is symmetrizable [86, 115, 116]. Yet its unperturbed part is symmetrizable by itself (because it describes a closed system too, namely, the medium absent a wave), so $\tau^{\alpha\beta}$ is also symmetrizable separately. On the other hand, since $\tau^{\alpha\beta}$ is proportional to the wave intensity, it is defined uniquely and, therefore, *must* be symmetric. This yields $\boldsymbol{\rho}=\boldsymbol{\vartheta}/c^2$, and

$$\boldsymbol{\mu}=\mathbf{x}\times\boldsymbol{\rho} \tag{74}$$

holds automatically [69, Sec. 32]. Also, since the *integral* energy-momentum of the whole WMS is defined uniquely [69, Sec. 32], and its a-dependent part is defined uniquely too, one can find $(\varepsilon/c,\boldsymbol{\rho})$ as the a-dependent part of the WMS *canonical* energy-momentum density. Given the WMS Lagrangian density, the latter can, in principle, be found straightforwardly in any specific problem [114]. However, the general answer is not informative (meaning that $\tau^{\alpha\beta}$ is by itself a somewhat artificial construct). Thus, below, we consider only the particular model of an isotropic medium, most popular in the AMC context, yet still refrain from specifying the wave nature.

B. Wave energy-momentum in isotropic medium

General case. — Consider an isotropic medium (such as gas, fluid, or plasma) comprised of elementary [136] particles or fluid elements whose dynamics absent a wave is described by some aggregate Lagrangian L. In the presence of a wave, the WMS Lagrangian is hence $L+\mathsf{L}$, where $\mathsf{L}=\int\mathfrak{L}\,dV$ is the wave Lagrangian. Assuming that particles contribute to L additively, the latter can be written as $\mathsf{L}=\mathsf{L}^{(0)}-\sum_\ell\Phi^{(\ell)}$, where $\mathsf{L}^{(0)}$ is independent of all particle velocities $\mathbf{u}^{(\ell)}$, and each of the so-called ponderomotive potentials $\Phi^{(\ell)}$ [104, 105], or dipole potentials [65, 66], depends on the specific $\mathbf{u}^{(\ell)}$ but not on other velocities. Omitting the index ℓ, we can write the canonical momentum of each particle as the sum of the mechanical part $\partial_{\mathbf{u}}L$ and the ponderomotive part $-\partial_{\mathbf{u}}\Phi$, also yielding the ponderomotive contribution to the particle canonical energy, $-\mathbf{u}\cdot\partial_{\mathbf{u}}\Phi$. (This energy should not be confused with the ponderomotive potential Φ itself, which a part of the *wave* canonical energy [137].) Thus, the densities of the ponderomotive momentum and energy stored in particles can be written as follows:

$$\Delta\boldsymbol{\rho}=-\sum_s n^{(s)}\langle\partial_{\mathbf{u}}\Phi\rangle^{(s)}, \tag{75}$$

$$\Delta\varepsilon=-\sum_s n^{(s)}\langle\mathbf{u}\cdot\partial_{\mathbf{u}}\Phi\rangle^{(s)}, \tag{76}$$

where the summation is taken over different species, $n^{(s)}$ are the (locally averaged) densities of those species, and angular brackets denote averaging over velocities within the corresponding ensembles.

Single-fluid model. — If a medium can be modeled as a single fluid (in particular meaning that kinetic effects are inessential, unlike, e.g., in hot plasma), one can simplify Eqs. (76) and (75) further, namely, as follows. First of all, notice that the velocities $\mathbf{u}$ of fluid elements are all equal to a single velocity $\mathbf{v}$, so Eqs. (75) and (76) become

$$\Delta\boldsymbol{\rho} = -n\,\partial_{\mathbf{v}}\Phi, \quad \Delta\varepsilon = \mathbf{v}\cdot\Delta\boldsymbol{\rho}. \tag{77}$$

It is hence convenient to rewrite Eqs. (77) in terms of Lorentz-invariant proper parameters of the medium [138]. Since Φ that enters here depends on the wave intensity, it must be gauge-invariant; thus, being (minus) the interaction Lagrangian of a single element, it transforms as $\Phi = \Phi'/\gamma$ [139], with primes *in this section* (Sec. V) denoting the medium rest frame, and $\gamma = (1 - v^2/c^2)^{-1/2}$. Also, $n = \gamma n'$, where n' is the proper density, correspondingly. Since the latter does not depend on $\mathbf{v}$, we then get $\Delta\boldsymbol{\rho} = -\partial_{\mathbf{v}}(n'\Phi') + \gamma^2\mathbf{v}n'\Phi'/c^2$. Further, let us denote

$$n'\Phi' = \mathfrak{L}' - \mathfrak{L}'^{(0)} \doteq \mathcal{U}', \tag{78}$$

where $\mathfrak{L}'^{(0)}$ is $\mathsf{L}'^{(0)}$ per unit volume, and introduce

$$\mathfrak{R} \doteq \frac{\gamma^2\mathbf{v}}{c^2}\,\mathcal{U}', \tag{79}$$

understood as the striction contribution (Sec. VI E). Since $\mathfrak{L}'^{(0)}$ is also independent of $\mathbf{v}$, one then can write

$$\Delta\boldsymbol{\rho} = \partial_{\mathbf{v}}\mathfrak{L}' + \mathfrak{R}. \tag{80}$$

Due to the fact that a Lagrangian density is a four-scalar, $\mathfrak{L}'$ that enters Eq. (80) can also be replaced with $\mathfrak{L}$. However, using $\mathfrak{L}'(a', k'_\mu)$ is preferable, because it cannot depend on $\mathbf{v}$ explicitly, but rather depends on it solely through a' and k'_μ. [Remember that the velocity derivative in Eq. (80) must be taken at fixed wave variables a and k_μ.] Due $\mathfrak{L}'_{a'} = 0$ [cf. Eq. (30)], we then get

$$\partial_{\mathbf{v}}\mathfrak{L}' = -(\partial_{\mathbf{v}}\Lambda^\nu{}_\mu)\,k_\nu\mathcal{J}'^\mu, \tag{81}$$

where we substituted the (covector) Lorentz transformation (2), i.e., $k'_\mu = \Lambda^\nu{}_\mu k_\nu$. On the other hand, $k_\nu = (\Lambda^{-1})^\lambda{}_\nu k'_\lambda$, so Eq. (81) can also be written as

$$\partial_{\mathbf{v}}\mathfrak{L}' = -\gamma\mathbf{G}^\lambda{}_\mu\mathcal{T}'^\mu_\lambda/c, \tag{82}$$

where we introduced a dimensionless matrix function

$$\mathbf{G}^\lambda{}_\mu(\mathbf{v}) \doteq (c/\gamma)(\Lambda^{-1})^\lambda{}_\nu\,(\partial_{\mathbf{v}}\Lambda^\nu{}_\mu). \tag{83}$$

As shown in Appendix, Eq. (82) is also equivalent to

$$\partial_{\mathbf{v}}\mathfrak{L}' = \gamma\mathrm{Tr}(\mathbf{G}\mathcal{T}')/c = \mathfrak{P} + \mathfrak{B}, \tag{84}$$

where the terms on the right-hand side are defined as

$$\mathfrak{P} = \gamma\hat{\boldsymbol{\Lambda}}\cdot\left(\frac{\mathcal{E}'\mathbf{v}'_g}{c^2} - \boldsymbol{\mathcal{P}}'\right), \tag{85}$$

$$\mathfrak{B} = \frac{\gamma^2}{\gamma+1}\left[\frac{\mathbf{v}}{c}\times\left(\frac{\mathbf{v}'_g}{c}\times\boldsymbol{\mathcal{P}}'\right)\right]. \tag{86}$$

Yet, $\mathbf{v}'_g$ is parallel to $\mathbf{k}'$ in isotropic medium, so $\mathfrak{B}$ vanishes, and we finally get

$$\Delta\boldsymbol{\rho} = \mathfrak{P} + \mathfrak{R}, \quad \Delta\varepsilon = \mathbf{v}\cdot(\mathfrak{P} + \mathfrak{R}). \tag{87}$$

C. Wave EMT in the single-fluid model

Within the single-fluid model, one can hence explicitly construct the complete kinetic EMT of a wave,

$$\tau^{\alpha\beta} = \Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu\tau'^{\mu\nu}, \tag{88}$$

which is done as follows.

Energy and momentum. — First of all, let us combine Eqs. (73) and (87) with Eq. (85) for $\mathfrak{P}$, Eq. (79) for $\mathfrak{R}$, and $\mathcal{E} = \omega\mathcal{I}$ and $\boldsymbol{\mathcal{P}} = \mathbf{k}\mathcal{I}$, as well as with

$$\mathcal{I} = \gamma\mathcal{I}'(1 + \mathbf{v}\cdot\mathbf{v}'_g/c^2), \tag{89}$$

where we employed the four-vector transformation properties of $\mathcal{J}^\alpha$. This yields

$$\varepsilon = \gamma^2\mathcal{E}' + \frac{\gamma\mathcal{E}'\mathbf{v}}{c^2}\cdot\left(\hat{\boldsymbol{\Lambda}}\cdot\mathbf{v}'_g + \frac{\omega}{\omega'}\,\mathbf{v}'_g\right) + \frac{\gamma^2v^2}{c^2}\,\mathcal{U}', \tag{90}$$

$$\boldsymbol{\rho} = \frac{\gamma\mathcal{E}'}{c^2}\left[\hat{\boldsymbol{\Lambda}}\cdot\mathbf{v}'_g + \gamma\mathbf{v} + \frac{\mathbf{k}}{\omega'}(\mathbf{v}\cdot\mathbf{v}'_g)\right] + \frac{\gamma^2\mathbf{v}}{c^2}\,\mathcal{U}'. \tag{91}$$

[Entering the numerator in Eq. (91) is actually $\mathbf{k}$, not $\mathbf{k}'$.] By taking $\mathbf{v} = 0$ here, we then get, in particular,

$$\varepsilon' = \mathcal{E}', \quad c\boldsymbol{\rho}' = \boldsymbol{\vartheta}'/c = \mathcal{E}'\mathbf{v}'_g/c, \tag{92}$$

also using that $\tau^{\alpha\beta}$ is symmetric in all reference frames.

Momentum flux density. — Since $\mathbf{k}'$ is the only designated direction in the medium rest frame, the (symmetric) momentum flux density $\hat{\boldsymbol{\pi}}'$ must be a linear superposition of $\mathbf{k}'\mathbf{k}'$ and $\hat{\mathbf{1}}'$, or, equivalently, $\hat{\boldsymbol{\pi}}' = \psi\,\mathbf{k}'\mathbf{v}'_g + \zeta\,\hat{\mathbf{1}}'$, where ψ and ζ are some coefficients. Combining this with Eqs. (88) and (92) and plus with, e.g., Eq. (90) for $\varepsilon \equiv \tau^{00}$, one readily obtains $\psi = \mathcal{I}'$ and $\zeta = \mathcal{U}'$; i.e.,

$$\hat{\boldsymbol{\pi}}' = \mathcal{E}'\,\mathbf{k}'\mathbf{v}'_g/\omega' + \mathcal{U}'\,\hat{\mathbf{1}}'. \tag{93}$$

(In particular, if $\mathbf{v}'_g = 0$, the term $\mathcal{U}'$ acts as the ponderomotive pressure; cf. Ref. [140].) Equation (91) then flows from Eq. (88) automatically; yet, Eq. (88) also gives

$$\hat{\boldsymbol{\pi}} = \frac{\omega'(\mathbf{k}'\cdot\mathbf{v}'_g\mathcal{E}')}{c^2|\mathbf{k}'|^2}\left(\frac{c^2\mathbf{k}\mathbf{k}}{\omega'^2} - \frac{\gamma^2\mathbf{v}\mathbf{v}}{c^2}\right) + \frac{\gamma\mathbf{v}\mathbf{v}}{c^2}\,\mathcal{E}' + \left(\hat{\mathbf{1}} + \frac{\gamma^2\mathbf{v}\mathbf{v}}{c^2}\right)\mathcal{U}'. \tag{94}$$

EMT and ponderomotive forces. — The wave kinetic EMT in isotropic fluid is hereby summarized as

$$\tau'^{\alpha\beta} = \begin{pmatrix} \mathcal{E} & \mathcal{E}\mathbf{v}_g/c \\ \mathcal{E}\mathbf{v}_g/c & \mathcal{E}\,\mathbf{k}\mathbf{v}_g/\omega + \mathcal{U}\,\hat{\mathbf{1}} \end{pmatrix}' \tag{95}$$

in the medium rest frame and is transformed to other frames via Eq. (88), as also spelled out in Eqs. (90), (91), and (94). In particular, if the flow velocity is negligible in a given frame, one can take $\Lambda^\alpha{}_\beta \approx \delta^\alpha_\beta$, so

$$\varepsilon \approx \mathcal{E}, \quad \boldsymbol{\rho} \approx \mathcal{E}\mathbf{v}_g/c^2, \quad \boldsymbol{\mu} \approx (\mathbf{x}\times\mathbf{v}_g)\mathcal{E}/c^2. \tag{96}$$

Finally, the ponderomotive four-force density $\bar{f}^\alpha$ that a wave imparts to a medium also can be calculated [113],

$$\bar{f}^\alpha = -\tau^{\alpha\beta}{}_{;\beta}, \tag{97}$$

whence, substituting $\bar{f}^\alpha = (\bar{w}/c, \bar{\mathbf{f}})$, one obtains

$$\bar{w} = -\partial_t \varepsilon - c^2 \nabla \cdot \boldsymbol{\rho}, \quad \bar{\mathbf{f}} = -\partial_t \boldsymbol{\rho} - \nabla \cdot \hat{\boldsymbol{\pi}}. \tag{98}$$

Here $\bar{w}$ has the meaning of the power density input into the medium, and $\bar{\mathbf{f}}$ is the usual three-force density.

These results, which rely essentially *only* on Eq. (11) and the single-isotropic-fluid approximation (without any reference to electromagnetism), represent a more concise and transparent version of those reported in Ref. [86] and generalize the latter to the case of waves of arbitrary nature; see also Sec. VI E.

D. Photon kinetic properties

The following energy, momentum, and angular momentum can now be assigned to a single photon:

$$h \doteq \varepsilon/\mathcal{N}, \quad \mathbf{p} \doteq \boldsymbol{\rho}/\mathcal{N}, \quad \mathbf{m} \doteq \boldsymbol{\mu}/\mathcal{N}. \tag{99}$$

Keep in mind, however, that these are merely quantities *per* photon rather than the momenta *of* a photon, in contrast with $(H, \mathbf{P}, \mathbf{M})$ that actually enter the photon motion equations [Eqs. (41) and (54)]. As a result, $(h, \mathbf{p}, \mathbf{m})$ do not enjoy the simple transformation properties of their canonical counterparts. In particular, the kinetic four-momentum $p^\alpha \doteq (h/c, \mathbf{p})$ is generally not a four-vector. One can easily check this, e.g., by using $p^\alpha = (cN)^{-1} \int \tau^{\alpha 0}\, dV$ with $\tau^{\alpha\beta}$ taken from Sec. V C and

$$\mathcal{N} = \gamma \mathcal{N}' (1 + \mathbf{v} \cdot \mathbf{v}'_g / c^2), \tag{100}$$

$$\mathcal{E}' = \hbar \omega' \mathcal{N}', \quad \boldsymbol{\mathcal{P}}' = \hbar \mathbf{k}' \mathcal{N}'. \tag{101}$$

Still, simple expressions are obtained from Eqs. (96) for isotropic fluid medium at rest; namely,

$$h \approx \hbar\omega, \quad \mathbf{p} \approx \hbar\omega \mathbf{v}_g / c^2, \quad \mathbf{m} \approx (\mathbf{x} \times \mathbf{v}_g) \hbar\omega / c^2. \tag{102}$$

Since here $\mathbf{v}_g$ is assumed to be parallel to $\mathbf{k}$, one also gets that $\mathbf{p}$ is parallel to $\mathbf{P}$, $\mathbf{m}$ is parallel to $\mathbf{M}$, and

$$p/P = m/M \approx 1/(n_p n_g). \tag{103}$$

These match the traditional Abraham formulas [1, 45], hence seen to hold for waves of arbitrary (not necessarily electromagnetic) nature. Yet it is clear now that the traditional formulas are, in fact, approximate and generally invalid for moving and hot media, in contrast with the Minkowski formulas for the canonical quantities [Eqs. (33) and (48)] that are more universal.

VI. LINEAR ELECTROMAGNETIC WAVES

Finally, let us apply the above results to illustrate how the properties of linear electromagnetic waves can be calculated explicitly within our general approach, without using Maxwell's equations for the wave envelope. Note also that similar calculations can be performed for nonlinear waves too, for which $\mathfrak{L}$ can be constructed from first principles as well. Some of nonlinear GO effects rendered transparent this way, in fact, may not be captured correctly by other existing theories. For an expanded discussion of these issues see Refs. [104–107].

A. Wave Lagrangian density

First, let us consider a nondissipative wave, as usual. The wave Lagrangian density can be expected in the form $\mathfrak{L} = \mathfrak{L}^{(0)} - \mathcal{U}$, where

$$\mathfrak{L}^{(0)} \doteq \frac{1}{16\pi} (\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{B}}^* \cdot \tilde{\mathbf{B}}) \tag{104}$$

is that in vacuum [104], $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are the electric and magnetic field envelopes, and $\mathcal{U}$ is the potential energy density of the wave-medium interaction [cf. Eq. (78)]. For linear, i.e., dipolar interaction, we can take [141, Secs. 4.2, 4.8, 5.7, 6.2]

$$\mathcal{U} = -\frac{1}{4} \operatorname{Re} \left(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{P}} + \tilde{\mathbf{B}}^* \cdot \tilde{\mathbf{M}} \right). \tag{105}$$

Here $\tilde{\mathbf{P}}$ is the electric dipole moment density (i.e., the polarization), and $\tilde{\mathbf{M}}$ is the magnetic dipole moment density (i.e., the magnetization); also, one factor 1/2 comes from the time-averaging, and the other 1/2 comes from the fact that $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{M}}$ are linear functions of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$, correspondingly. Now let us introduce $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{H}}$ via

$$\tilde{\mathbf{D}} \doteq \tilde{\mathbf{E}} + 4\pi \tilde{\mathbf{P}} \doteq \hat{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}}, \tag{106}$$

$$\tilde{\mathbf{B}} \doteq \tilde{\mathbf{H}} + 4\pi \tilde{\mathbf{M}} \doteq \hat{\boldsymbol{\mu}} \cdot \tilde{\mathbf{H}}, \tag{107}$$

assuming that the permittivity tensor $\hat{\boldsymbol{\epsilon}}$ and the permeability tensor $\hat{\boldsymbol{\mu}}$ (not to be confused with the kinetic angular momentum density $\boldsymbol{\mu}$) are Hermitian so the assumption of zero dissipation be satisfied. One gets then [10, 86]

$$\mathfrak{L} = \frac{1}{16\pi} \left(\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{B}}^* \cdot \hat{\boldsymbol{\mu}}^{-1} \cdot \tilde{\mathbf{B}} \right) \tag{108}$$

(here $\hat{\boldsymbol{\mu}}^{-1}$ is the tensor inverse to $\hat{\boldsymbol{\mu}}$), also meaning that

$$\mathcal{U} = -\frac{1}{16\pi} \left[\tilde{\mathbf{E}}^* \cdot (\hat{\boldsymbol{\epsilon}} - \hat{\mathbf{1}}) \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{B}}^* \cdot (\hat{\boldsymbol{\mu}}^{-1} - \hat{\mathbf{1}}) \cdot \tilde{\mathbf{B}} \right]. \tag{109}$$

In agreement with Refs. [104, 105], this implies assigning the following ponderomotive potentials to particles (or fluid elements) comprising the medium:

$$\Phi = -\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\alpha}} \cdot \tilde{\mathbf{E}}/4 - \tilde{\mathbf{B}}^* \cdot \hat{\boldsymbol{\beta}} \cdot \tilde{\mathbf{B}}/4, \tag{110}$$

where $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ are the particle electric and magnetic polarizabilities [66], and

$$\hat{\boldsymbol{\epsilon}} = \hat{\mathbf{1}} + \sum_s 4\pi n^{(s)} \langle \hat{\boldsymbol{\alpha}} \rangle^{(s)}, \tag{111}$$

$$\hat{\boldsymbol{\mu}}^{-1} = \hat{\mathbf{1}} - \sum_s 4\pi n^{(s)} \langle \hat{\boldsymbol{\beta}} \rangle^{(s)}. \tag{112}$$

B. Parametrization and dispersion

Remember that there is a freedom in defining a, so there are various options for how to parameterize the wave Lagrangian density. First, let us consider $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{E}}^*$ as independent vector fields; i.e., $a = (\tilde{\mathbf{E}}, \tilde{\mathbf{E}}^*)$. In this case, it is convenient to write

$$\mathfrak{L}^{(0)} = \frac{1}{16\pi} \left(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}} - \frac{c^2}{\omega^2} |\mathbf{k} \times \tilde{\mathbf{E}}|^2 \right) \tag{113}$$

(where we used that $\tilde{\mathbf{B}} = c\mathbf{k} \times \tilde{\mathbf{E}}/\omega$) and

$$\mathfrak{L} = \frac{1}{16\pi} \left[\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}} - \frac{c^2}{\omega^2} (\mathbf{k} \times \tilde{\mathbf{E}}^*) \cdot \hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}}) \right], \tag{114}$$

correspondingly. Using that

$$\begin{aligned}(\mathbf{k} \times \tilde{\mathbf{E}}^*) \cdot \hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}}) &= \\ - (\tilde{\mathbf{E}}^* \times \mathbf{k}) \cdot \hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}}) &= \\ - \tilde{\mathbf{E}}^* \cdot \{\mathbf{k} \times [\hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}})]\},\end{aligned} \tag{115}$$

one can further rewrite Eq. (114) as follows:

$$\mathfrak{L} = \frac{\tilde{\mathbf{E}}^*}{16\pi} \cdot \left\{ \hat{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}} + \frac{c^2}{\omega^2} \mathbf{k} \times \left[\hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}}) \right] \right\}. \tag{116}$$

Then, varying $\mathfrak{L}$ with respect to $\tilde{\mathbf{E}}^*$ yields the following dispersion relation:

$$\hat{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}} + \frac{c^2}{\omega^2} \mathbf{k} \times \left[\hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \tilde{\mathbf{E}}) \right] = 0, \tag{117}$$

in agreement with Maxwell's equations [142, Sec. 3.4]. Similarly, varying $\mathfrak{L}$ with respect to $\tilde{\mathbf{E}}$ yields the complex-conjugate equation.

Alternatively, if the polarization vector $\mathbf{e}$ is prescribed (or considered as an independent field), one can as well introduce a scalar amplitude instead, say, $a = |\tilde{\mathbf{E}}|$. This yields $\mathfrak{L} = \mathfrak{D}(\omega, \mathbf{k}) a^2$, with $\mathfrak{D}(\omega, \mathbf{k})$ given by

$$\mathfrak{D} = \frac{1}{16\pi} \left[\mathbf{e}^* \cdot \hat{\boldsymbol{\epsilon}} \cdot \mathbf{e} - \frac{c^2}{\omega^2} (\mathbf{k} \times \mathbf{e}^*) \cdot \hat{\boldsymbol{\mu}}^{-1} \cdot (\mathbf{k} \times \mathbf{e}) \right]. \tag{118}$$

The dispersion relation that follows [Eq. (30)] is Eq. (117) multiplied by $\mathbf{e}^*$.

C. Wave action and canonical EMT

The action density $\mathcal{I}$ is now obtained straightforwardly by differentiating $\mathfrak{L}$ [e.g., Eq. (114)] with respect to ω:

$$\mathcal{I} = \frac{1}{16\pi} \left[\tilde{\mathbf{E}}^* \cdot \hat{\epsilon}_\omega \cdot \tilde{\mathbf{E}} + \frac{2}{\omega} \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}} - \tilde{\mathbf{B}}^* \cdot (\hat{\boldsymbol{\mu}}^{-1})_\omega \cdot \tilde{\mathbf{B}} \right],$$

where we used $\tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}} = \tilde{\mathbf{B}}^* \cdot \tilde{\mathbf{H}}$, due to $\hat{\boldsymbol{\mu}}$ being Hermitian. From $\mathfrak{L} = 0$ [Eq. (30)], one also has

$$\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} = \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}}. \tag{119}$$

Thus, $\mathcal{I} = \mathcal{I}^{(E)} + \mathcal{I}^{(B)}$, where

$$\mathcal{I}^{(E)} = \frac{1}{16\pi} \left[\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\epsilon}}_\omega \cdot \tilde{\mathbf{E}} + \frac{1}{\omega} \tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\epsilon}} \cdot \tilde{\mathbf{E}} \right], \tag{120}$$

$$\mathcal{I}^{(B)} = \frac{1}{16\pi} \left[\frac{1}{\omega} \tilde{\mathbf{H}}^* \cdot \hat{\boldsymbol{\mu}} \cdot \tilde{\mathbf{H}} - \tilde{\mathbf{B}}^* \cdot (\hat{\boldsymbol{\mu}}^{-1})_\omega \cdot \tilde{\mathbf{B}} \right]. \tag{121}$$

One can further substitute

$$\begin{aligned}\tilde{\mathbf{B}}^* \cdot (\hat{\boldsymbol{\mu}}^{-1})_\omega \cdot \tilde{\mathbf{B}} &= \tilde{\mathbf{B}}^* \cdot (\hat{\boldsymbol{\mu}}^{-1})_\omega \cdot \hat{\boldsymbol{\mu}} \cdot \tilde{\mathbf{H}} = \\ &= -\tilde{\mathbf{B}}^* \cdot \hat{\boldsymbol{\mu}}^{-1} \cdot \hat{\boldsymbol{\mu}}_\omega \cdot \tilde{\mathbf{H}} = -\tilde{\mathbf{H}}^* \cdot \hat{\boldsymbol{\mu}}_\omega \cdot \tilde{\mathbf{H}},\end{aligned} \tag{122}$$

where we used $(\hat{\boldsymbol{\mu}}^{-1} \cdot \hat{\boldsymbol{\mu}})_\omega \equiv 0$. Therefore,

$$\mathcal{I} = \frac{1}{16\pi\omega} \left[\tilde{\mathbf{E}}^* \cdot (\omega\hat{\boldsymbol{\epsilon}})_\omega \cdot \tilde{\mathbf{E}} + \tilde{\mathbf{H}}^* \cdot (\omega\hat{\boldsymbol{\mu}})_\omega \cdot \tilde{\mathbf{H}} \right], \tag{123}$$

whence the elements of the wave canonical EMT [Eq. (26)] are readily found; namely,

$$\mathcal{E} = \omega\mathcal{I}, \quad \boldsymbol{\mathcal{Q}} = \mathbf{v}_g \omega \mathcal{I}, \quad \boldsymbol{\mathcal{P}} = \mathbf{k}\mathcal{I}, \quad \hat{\mathbf{\Pi}} = \mathbf{k}\mathbf{v}_g \mathcal{I}. \tag{124}$$

D. Dissipative waves

In the presence of dissipation, the dispersion relation flowing from Maxwell's equations is similar to that in Sec. VI B. Namely, it can be written as $\mathfrak{D}(\Omega, \mathbf{K}) = 0$, where $\mathfrak{D}$ has the same form as in Eq. (118), yet now with

$$\hat{\boldsymbol{\epsilon}} = \hat{\boldsymbol{\epsilon}}' + i\hat{\boldsymbol{\epsilon}}'', \quad \hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}}' + i\hat{\boldsymbol{\mu}}''. \tag{125}$$

where $\hat{\boldsymbol{\epsilon}}'$ and $\hat{\boldsymbol{\mu}}'$ are Hermitian, and $i\hat{\boldsymbol{\epsilon}}''$ and $i\hat{\boldsymbol{\mu}}''$ are anti-Hermitian. Using that

$$(\hat{\boldsymbol{\mu}}' + i\hat{\boldsymbol{\mu}}'')^{-1} \approx \hat{\boldsymbol{\mu}}'^{-1} - i\hat{\boldsymbol{\mu}}'^{-1} \cdot \hat{\boldsymbol{\mu}}'' \cdot \hat{\boldsymbol{\mu}}'^{-1}, \tag{126}$$

we can hence write, for $\mathfrak{D}$ evaluated at real $(\omega, \mathbf{k})$, that $\mathfrak{D} = \mathfrak{D}' + i\mathfrak{D}''$, where $\mathfrak{D}'$ and $\mathfrak{D}''$ are real and given by

$$\mathfrak{D}' = \frac{1}{16\pi} \left[\mathbf{e}^* \cdot \hat{\boldsymbol{\epsilon}}' \cdot \mathbf{e} - \frac{c^2}{\omega^2} (\mathbf{k} \times \mathbf{e}^*) \cdot \hat{\boldsymbol{\mu}}'^{-1} \cdot (\mathbf{k} \times \mathbf{e}) \right],$$

$$\mathfrak{D}'' = \frac{1}{16\pi} \left[\mathbf{e}^* \cdot \hat{\boldsymbol{\epsilon}}'' \cdot \mathbf{e} + \frac{c^2}{\omega^2} (\mathbf{k} \times \mathbf{e}^*) \cdot \hat{\boldsymbol{\mu}}'^{-1} \cdot \hat{\boldsymbol{\mu}}'' \cdot \hat{\boldsymbol{\mu}}'^{-1} \cdot (\mathbf{k} \times \mathbf{e}) \right].$$

According to Sec. IV D, we can infer $\mathcal{I}$ directly from Eq. (123) by replacing $\mathfrak{D}$ with $\mathfrak{D}'$, so

$$\mathcal{I} = \frac{1}{16\pi\omega}\left[\tilde{\mathbf{E}}^* \cdot (\omega\hat{\boldsymbol{\epsilon}}')_\omega \cdot \tilde{\mathbf{E}} + \tilde{\mathbf{H}}^* \cdot (\omega\hat{\boldsymbol{\mu}}')_\omega \cdot \tilde{\mathbf{H}}\right]. \quad (127)$$

Then the known formula [70, Sec. 80] for the energy density is recovered from $\mathcal{E} = \omega\mathcal{I}$. Other local properties of the wave are found from Eqs. (32) and (48), the dissipation rate Γ is found from Eq. (65), and Eq. (71) yields

$$\imath_{\text{loss}} = \frac{1}{8\pi}\left(\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\epsilon}}'' \cdot \tilde{\mathbf{E}} + \tilde{\mathbf{H}}^* \cdot \hat{\boldsymbol{\mu}}'' \cdot \tilde{\mathbf{H}}\right), \quad (128)$$

where we substituted $\varkappa = 2$, since $A = a^2$. The expression for the dissipation power density, $w_{\text{loss}} = \omega\imath_{\text{loss}}$, hence also agrees with the known formula [70, Sec. 80].

E. Kinetic EMT

Assuming that dissipation is negligible and the medium is isotropic, the wave kinetic EMT, as well as the kinetic angular momentum, can be found using results obtained in Sec. V. Specifically, one can use Eqs. (75) and (76) in the general case, substituting Eq. (110) for Φ. In the single-fluid approximation, Eqs. (90), (91), and (94) can be used in combination with Eqs. (123) and (124) taken in the medium rest frame. Due to Eq. (109), one can also take, in particular,

$$\mathfrak{R} = -\frac{\gamma^2\mathbf{v}}{c^2}\left(n\,\frac{\partial\epsilon}{\partial n}\,\frac{|\tilde{\mathbf{E}}|^2}{16\pi} - n\,\frac{\partial\mu^{-1}}{\partial n}\,\frac{|\tilde{\mathbf{B}}|^2}{16\pi}\right), \quad (129)$$

where the expression in parenthesis (equal to the interaction-Lagrangian density $-\mathcal{U}$) is Lorentz-invariant. Hence $\mathfrak{R}$ can be attributed to electrostriction and magnetostriction [86, 143]. Besides, one can show that Eqs. (170) and (171) of Ref. [86], derived there from different considerations, are recovered from our Eqs. (90), (91), and (94) as a special case. The proof is straightforward and will not be presented here.

F. Special case $\hat{\boldsymbol{\mu}} = \hat{\mathbf{1}}$

Since $\tilde{\mathbf{B}}$ is proportional to $\tilde{\mathbf{E}}$, one usually can define the high-frequency medium-response tensors $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\mu}}$ such that $\hat{\boldsymbol{\mu}} = 1$ (in a selected frame of reference). As this is done often, e.g., in plasma physics [142], let us also simplify some of the above expressions for this particular case. First of all, Eq. (108) yields

$$\mathfrak{L} = \frac{1}{16\pi}\left[\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\epsilon}}' \cdot \tilde{\mathbf{E}} - \frac{c^2}{\omega^2}\,|\mathbf{k}\times\tilde{\mathbf{E}}|^2\right], \quad (130)$$

or $\mathfrak{L} = \mathfrak{L}^{(0)} + \tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\chi}}' \cdot \tilde{\mathbf{E}}/(16\pi)$, where $\mathfrak{L}^{(0)}$ is the vacuum Lagrangian [Eq. (113)], and we introduced the electric susceptibility $\hat{\boldsymbol{\chi}} \doteq \hat{\boldsymbol{\epsilon}} - 1$. Then the wave energy is

$$\mathcal{E} = \frac{1}{16\pi}\left[\tilde{\mathbf{E}}^* \cdot (\omega\hat{\boldsymbol{\epsilon}}')_\omega \cdot \tilde{\mathbf{E}} + |\tilde{\mathbf{B}}|^2\right], \quad (131)$$

or, equivalently [due to $\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\epsilon}}' \cdot \tilde{\mathbf{E}} = |\tilde{\mathbf{B}}|^2$; cf. Eq. (119)],

$$\mathcal{E} = \frac{1}{16\pi\omega}\,\tilde{\mathbf{E}}^* \cdot (\omega^2\hat{\boldsymbol{\epsilon}}')_\omega \cdot \tilde{\mathbf{E}}. \quad (132)$$

Also, as usual, the canonical momentum density equals

$$\boldsymbol{\mathcal{P}} = \mathbf{k}\mathcal{E}/\omega. \quad (133)$$

One can show, using Eq. (117), that the latter is just a more concise form of the corresponding expression in Ref. [144]. Contrary to Ref. [145], calculated there is thus not the total, but only the canonical momentum (and the canonical energy) of the wave; see also Refs. [146, 147].

Following Ref. [142], let us also separate the energy flux $\boldsymbol{\mathcal{Q}}$ into the electromagnetic part and the kinetic part. Specifically, using $\boldsymbol{\mathcal{Q}} = -\omega\mathfrak{L}_{\mathbf{k}}$, one can write it as $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{K}}$, where $\boldsymbol{\mathcal{S}} = -\omega\mathfrak{L}^{(0)}_{\mathbf{k}}$, and

$$\boldsymbol{\mathcal{K}} = -\frac{\omega}{16\pi}\,\tilde{\mathbf{E}}^* \cdot \hat{\boldsymbol{\chi}}'_{\mathbf{k}} \cdot \tilde{\mathbf{E}}. \quad (134)$$

The latter is recognized as the energy flux density caused by the presence of the medium [142, Chap. 4], whereas

$$\begin{aligned}
\boldsymbol{\mathcal{S}} &= \frac{c^2}{16\pi\omega}\Big\{(\underline{\mathbf{k}}\times\tilde{\mathbf{E}}^*)\cdot(\mathbf{k}\times\tilde{\mathbf{E}}) + (\mathbf{k}\times\tilde{\mathbf{E}}^*)\cdot(\underline{\mathbf{k}}\times\tilde{\mathbf{E}})\Big\}_{\mathbf{k}} \\
&= \frac{c^2}{16\pi\omega}\Big\{\underline{\mathbf{k}}\cdot\big[\tilde{\mathbf{E}}^*\times(\mathbf{k}\times\tilde{\mathbf{E}})\big] + \underline{\mathbf{k}}\cdot\big[\tilde{\mathbf{E}}\times(\mathbf{k}\times\tilde{\mathbf{E}}^*)\big]\Big\}_{\mathbf{k}} \\
&= \frac{c^2}{16\pi\omega}\Big\{\tilde{\mathbf{E}}^*\times(\mathbf{k}\times\tilde{\mathbf{E}}) + \tilde{\mathbf{E}}\times(\mathbf{k}\times\tilde{\mathbf{E}}^*)\Big\} \\
&= \frac{c}{8\pi}\,\mathrm{Re}\,(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*)
\end{aligned} \quad (135)$$

is the time-averaged Poynting vector, i.e., the "vacuum part" of $\boldsymbol{\mathcal{Q}}$. [Here we substituted Eq. (113) and used underlining to specify where the differentiation applies.] Recalling that $\boldsymbol{\mathcal{Q}} = \mathcal{E}\mathbf{v}_g$, one then also recovers the known formula [142, Chap. 4]

$$\mathbf{v}_g = (\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{K}})/\mathcal{E}, \quad (136)$$

and, in particular, for a single-fluid dielectric *at rest*,

$$\varepsilon = \mathcal{E}, \quad \boldsymbol{\rho} = (\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{K}})/c^2, \quad \boldsymbol{\mu} = [\mathbf{x}\times(\boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{K}})]/c^2.$$

For example, for an electromagnetic wave in vacuum, the above results yield $\omega^2 = c^2k^2$, so $\mathbf{v}_g = c^2\mathbf{k}/\omega = c\mathfrak{n}$, where $\mathfrak{n} \doteq \mathbf{k}/k$. Then $\mathcal{E} = \tilde{E}^2/(8\pi)$; $c^2\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ are equal to each other (so the EMT is symmetric, and canonical quantities coincide with kinetic quantities) and $\boldsymbol{\mathcal{S}} = \mathfrak{n}c\mathcal{E}$; also, $\hat{\boldsymbol{\Pi}} = \mathfrak{n}\mathfrak{n}\,\mathcal{E}$ equals minus the time-averaged Maxwell stress tensor. Thus, in this case, the wave EMT coincides with the electromagnetic stress-energy tensor [69, Sec. 32]. Besides, $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mu}$ are both equal to $\mathbf{x}\times\boldsymbol{\mathcal{S}}/c^2$, in agreement with the traditional definition of the wave angular momentum density in vacuum [132].

VII. CONCLUSIONS

In this paper, we restate classical GO axiomatically within the field-theoretical approach, also extended to account for dissipation. The concept of a photon in a dispersive medium is introduced, and photon properties are calculated unambiguously. In particular, the canonical and kinetic momenta and angular momenta carried by a photon, as well as the two corresponding EMTs, are derived straightforwardly from first principles of Lagrangian mechanics. The Abraham-Minkowski controversy pertaining to the definitions of these quantities is thereby resolved, and corrections to the traditional formulas for the photon kinetic quantities are found. An application of axiomatic GO to electromagnetic waves is also presented, yet merely as an example, whereas our main results apply to waves of arbitrary nature.

The work was supported by the NNSA SSAA Program through DOE Research Grant No. DE274-FG52-08NA28553 and by the U.S. DOE through Contract No. DE-AC02-09CH11466.

APPENDIX A: AUXILIARY FUNCTION $\mathbf{G}^{\lambda}{}_{\mu}$

Here, we summarize the properties of a dimensionless matrix function $\mathbf{G}^{\lambda}{}_{\mu}$ introduced in Eq. (83). First of all, notice an obvious equality

$$(\Lambda^{-1})^{\lambda}{}_{\nu}(\mathbf{v}) = \Lambda^{\lambda}{}_{\nu}(-\mathbf{v}), \tag{A1}$$

which can also be checked by confirming that

$$\Lambda^{\mu}{}_{\lambda}(-\mathbf{v})\Lambda^{\lambda}{}_{\nu}(\mathbf{v}) = \delta^{\mu}_{\nu}. \tag{A2}$$

Then a direct calculation yields

$$G^{0}{}_{0l} = 0, \tag{A3}$$

$$G^{i}{}_{0l} = \Lambda^{i}{}_{l}, \quad G^{0}{}_{il} = \eta_{ij}\Lambda^{j}{}_{l}, \tag{A4}$$

$$G^{i}{}_{jl} = (\delta^{i}_{l} v_j - \eta_{jl} v^i)(\gamma/c)/(\gamma+1), \tag{A5}$$

where we introduced the notation

$$G^{\lambda}{}_{\mu l} \equiv (\mathbf{G}^{\lambda}{}_{\mu})_l \equiv (c/\gamma)(\Lambda^{-1})^{\lambda}{}_{\nu}\,(\partial\Lambda^{\nu}{}_{\mu}/\partial v^l). \tag{A6}$$

(In particular, notice that the three l-components, $G^{\nu}{}_{\mu l}$, at $\mathbf{v} = 0$ happen to be the well-known Lorentz boost generators.) Let us now *define* the function

$$G_{\nu\mu l} \doteq g_{\nu\lambda} G^{\lambda}{}_{\mu l}. \tag{A7}$$

Due to Eqs. (16), one finds the latter to be

$$G_{00l} = 0, \tag{A8}$$

$$G_{0il} = -G_{i0l} = -\eta_{ij}\Lambda^{j}{}_{l}, \tag{A9}$$

$$G_{ijl} = (\eta_{il} v_j - \eta_{jl} v_i)(\gamma/c)/(\gamma+1), \tag{A10}$$

so, in particular,

$$\mathbf{G}_{\nu\mu} = -\mathbf{G}_{\mu\nu}, \quad (\mathbf{G}_{\mu\nu})_l \equiv G_{\mu\nu l}. \tag{A11}$$

Hence Eq. (82) becomes

$$\begin{aligned}\partial_{\mathbf{v}}\mathfrak{L}' &= -\gamma\mathbf{G}^{\lambda}{}_{\mu} g_{\lambda\nu}\mathcal{T}'^{\nu\mu}/c = -\gamma g_{\nu\lambda}\mathbf{G}^{\lambda}{}_{\mu}\mathcal{T}'^{\nu\mu}/c \\ &= -\gamma\mathbf{G}_{\nu\mu}\mathcal{T}'^{\nu\mu}/c = \gamma\mathbf{G}_{\mu\nu}\mathcal{T}'^{\nu\mu}/c,\end{aligned} \tag{A12}$$

which is exactly Eq. (84), where we substituted Eq. (A8) and introduced

$$\mathfrak{P}_l \doteq (\gamma/c)(G_{i0l}\mathcal{T}'^{0i} + G_{0il}\mathcal{T}'^{i0}), \quad \mathfrak{B}_l \doteq (\gamma/c) G_{ijl}\mathcal{T}'^{ji}.$$

Finally, due to Eqs. (A9)-(A10),

$$\begin{aligned}\mathfrak{P}_l &= \gamma(\eta_{ij}\Lambda^{j}{}_{l}\mathcal{E}' v'^{i}_{g}/c^2 - \eta_{ij}\Lambda^{j}{}_{l}\mathcal{P}'^{i}) \\ &= \gamma\Lambda^{j}{}_{l}(\mathcal{E}' v'_{gj}/c^2 - \mathcal{P}'_j) = [\gamma\hat{\mathbf{\Lambda}}\cdot(\mathcal{E}'\mathbf{v}'_g/c^2 - \boldsymbol{\mathcal{P}}')]_l,\end{aligned}$$

$$\begin{aligned}&[(\gamma+1)c^2/\gamma^2]\mathfrak{B}_l \\ &\quad = (\eta_{il} v_j - \eta_{jl} v_i)\mathcal{P}'^{j} v'^{i}_{g} = (v'_{gl} v_j \mathcal{P}'^{j} - \mathcal{P}'_l v_i v'^{i}_{g}) \\ &\quad = [\mathbf{v}'_g(\mathbf{v}\cdot\boldsymbol{\mathcal{P}}') - \boldsymbol{\mathcal{P}}'(\mathbf{v}\cdot\mathbf{v}'_g)]_l = [\mathbf{v}\times(\mathbf{v}'_g\times\boldsymbol{\mathcal{P}}')]_l,\end{aligned}$$

whence Eqs. (85) and (86) readily follow.

[1] S. M. Barnett, Phys. Rev. Lett. **104**, 070401 (2010).
[2] G. L. J. A. Rikken and B. A. van Tiggelen, arXiv:1204.6617 (2012).
[3] M. E. Crenshaw and T. B. Bahder, arXiv:1203.5018 (2012).
[4] C. Wang, arXiv:1202.2575 (2012).
[5] T. Ramos, G. F. Rubilar, and Y. N. Obukhov, Phys. Lett. A **375**, 1703 (2011).
[6] M. E. Crenshaw and T. B. Bahder, Opt. Commun. **284**, 2460 (2011).
[7] Y. He, J. Shen, and S. He, Prog. Electromag. Res. **116**, 81 (2011).
[8] X. Liu and G. Zhang, Opt. Comm. **284**, 4546 (2011).
[9] I. Brevik and S. A. Ellingsen, Ann. Phys. **326**, 754 (2011).
[10] T. G. Philbin, Phys. Rev. A **83**, 013823 (2011).
[11] J. L. Jiménez, I. Campos, and M. A. López-Mariño, Eur. J. Phys. **32**, 739 (2011).
[12] J. Yu, C. Chen, Y. Zhai, Z. Chen, J. Zhang, L. Wu, F. Huang, and Yi Xiao, Opt. Express **19**, 25263 (2011).
[13] T. K. Nakamura, arXiv:1112.2570 (2011).
[14] F. Ravndal, arXiv:1107.5074 (2011).
[15] I. Ojima and H. Saigo, arXiv:1101.5782 (2011).
[16] C. Wang, arXiv:1106.1163 (2011).
[17] W.-Z. Zhang, P. Zhang, Ru-Q. Wang, and Wu-M. Liu, arXiv:1012.2712 (2011).
[18] M. Mansuripur and A. R. Zakharian, Opt. Comm. **283**, 3557 (2010).

[19] A. Shevchenko and B. J. Hoenders, New J. Phys. **12**, 053020 (2010).
[20] P. L. Saldanha, Opt. Express **18**, 2258 (2010).
[21] D. H. Bradshaw, Z. Shi, R. W. Boyd, and P. W. Milonni, Opt. Comm. **283**, 650 (2010).
[22] V. G. Veselago and V. V. Shchavlev, Uspekhi Fiz. Nauk **180**, 331 (2010) [Phys. Uspekhi **53**, 317 (2010)].
[23] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Phys. Rev. A **79**, 023813 (2009).
[24] E. A. Hinds and S. M. Barnett, Phys. Rev. Lett. **102**, 050403 (2009).
[25] M. Mansuripur, Phys. Rev. Lett. **103**, 019301 (2009).
[26] Z. Y. Wang, arXiv:0909.1856 (2009).
[27] M. J. Padgett, Opt. Express **16**, 20864 (2008).
[28] M. G. Scullion and S. M. Barnett, J. Mod. Phys. **55**, 2301 (2008).
[29] B. A. Kemp, J. A. Kong, and T. M. Grzegorczyk, Phys. Rev. A **75**, 053810 (2007).
[30] T. Dereli, J. Gratus, and R. W. Tucker, J. Phys. A **40**, 5695 (2007).
[31] T. Dereli, J. Gratus, and R. W. Tucker, Phys. Lett. A **361**, 190 (2007).
[32] M. Mansuripur, Opt. Express **15**, 2677 (2007).
[33] U. Leonhardt, Phys. Rev. A **73**, 032108 (2006).
[34] U. Leonhardt, Nature **444**, 823 (2006).
[35] S. Stallinga, Phys. Rev. E **73**, 026606 (2006).
[36] S. Stallinga, Opt. Express **14**, 1286 (2006).
[37] R. Loudon, S. M. Barnett, and C. Baxter, Phys. Rev. A **71**, 063802 (2005).
[38] M. Mansuripur, Opt. Express **13**, 2245 (2005).
[39] P. W. Milonni and R. W. Boyd, Laser Phys. **15**, 1432 (2005).
[40] J. C. Garrison and R. Y. Chiao, Phys. Rev. A **70**, 053826 (2004).
[41] A. Feigel, Phys. Rev. Lett. **92**, 020404 (2004).
[42] M. A. López-Mariño and J. L. Jiménez, Found. Phys. Lett. **17**, 1 (2004).
[43] M. Mansuripur, Opt. Express **12**, 5375 (2004).
[44] B. A. van Tiggelen and G. L. J. A. Rikken, Phys. Rev. Lett. **93**, 268903 (2004).
[45] M. Padgett, S. M. Barnett, and R. Loudon, J. Mod. Opt. **50**, 1555 (2003).
[46] Y. N. Obukhov and F. W. Hehl, Phys. Lett. A **311**, 277 (2003).
[47] R. Loudon, J. Mod. Opt. **49**, 821 (2002).
[48] R. Loudon, L. Allen, and D. F. Nelson, Phys. Rev. E **55**, 1071 (1997).
[49] M. Labardi, G. C. La Rocca, F. Mango, F. Bassani, and M. Allegrinia, J. Vac. Sci. Technol. B **14**, 868 (1996).
[50] M. Kristensen and J. P. Woerdman, Phys. Rev. Lett. **72**, 2171 (1994).
[51] C. Baxter, M. Babiker, and R. Loudon, Phys. Rev. A **47**, 1278 (1993).
[52] D. F. Nelson, Phys. Rev. A **44**, 3985 (1991).
[53] Zh.-Y. Wang, P.-Y. Wang, and Y.-R. Xu, Optik **122**, 1994 (2011).
[54] W. She, J. Yu, and R. Feng, Phys. Rev. Lett. **101**, 243601 (2008).
[55] G. K. Campbell, A. E. Leanhardt, J. Mun, M. Boyd, E. W. Streed, W. Ketterle, and D. E. Pritchard, Phys. Rev. Lett. **94**, 170403 (2005).
[56] S. M. Barnett and R. Loudon, Phil. Trans. R. Soc. A **368**, 927 (2010).
[57] B. A. Kemp, J. Appl. Phys. **109**, 111101 (2011).
[58] P. W. Milonni and R. W. Boyd, Adv. Opt. Photon. **572**, 519 (2010).
[59] C. Baxter and R. Loudon, J. Mod. Opt. **57**, 830 (2010).
[60] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Rev. Mod. Phys. **79**, 1197 (2007).
[61] R. Loudon, Fortschr. Phys. **52**, 1134 (2004).
[62] I. Brevik, Phys. Rep. **52**, 133 (1979).
[63] S. W. McDonald and A. N. Kaufman, Phys. Rev. A **32**, 1708 (1985).
[64] I. Y. Dodin, arXiv:1107.2852 (2011), in Proceedings of the 30th ICPIG, AW1-413 (Belfast, UK, 2011).
[65] I. Y. Dodin and N. J. Fisch, Phys. Rev. E **79**, 026407 (2009).
[66] I. Y. Dodin and N. J. Fisch, Phys. Lett. A **374**, 3472 (2010).
[67] I. Y. Dodin, A. I. Zhmoginov, and N. J. Fisch, Phys. Lett. A **372**, 6094 (2008).
[68] I. Y. Dodin and N. J. Fisch, Phys. Plasmas **14**, 055901 (2007).
[69] L. D. Landau and E. M. Lifshitz, *The classical theory of fields* (Pergamon Press, New York, 1971).
[70] L. D. Landau and E. M. Lifshitz, *Electrodynamics of continuous media* (Pergamon Press, New York, 1993).
[71] Yu. A. Kravtsov and Yu. I. Orlov, *Geometrical optics of inhomogeneous media* (Springer-Verlag, New York, 1990).
[72] S. Weinberg, Phys. Rev. **126**, 1899 (1962).
[73] Yu. A. Kravtsov, L. A. Ostrovsky, and N. S. Stepanov, Proc. IEEE **62**, 1492 (1974).
[74] I. B. Bernstein, Phys. Fluids **18**, 320 (1975).
[75] I. B. Bernstein and D. E. Baldwin, Phys. Fluids **20**, 116 (1977).
[76] M. Bornatici and Yu. A. Kravtsov, Plasma Phys. Control. Fusion **42**, 255 (2000).
[77] M. Bornatici and O. Maj, Plasma Phys. Control. Fusion **45**, 1511 (2003).
[78] Also, in their general form, GO equations are usually more robust and transparent compared to equations derived *ad hoc*. For examples see, e.g., Refs. [140, 148, 149].
[79] P. A. Sturrock, Ann. Phys. **4**, 306 (1958).
[80] P. A. Sturrock, J. Appl. Phys. **31**, 2052 (1960).
[81] R. A. Toupin, Arch. Ration. Mech. Analys. **5**, 440 (1960).
[82] P. A. Sturrock, Ann. Phys. **15**, 250 (1961).
[83] G. B. Whitham, J. Fluid Mech. **22**, 273 (1965).
[84] G. B. Whitham, *Linear and nonlinear waves* (Wiley, New York, 1974).
[85] J. P. Dougherty, J. Plasma Phys. **4**, 761 (1970).
[86] R. L. Dewar, Austral. J. Phys. **30**, 533 (1977).
[87] M. J. Lighthill, J. Inst. Math. Appl. **1**, 269 (1965).
[88] C. J. R. Garrett, Proc. Roy. Soc. A **299**, 26 (1967).
[89] F. P. Bretherton and C. J. R. Garrett, Proc. Roy. Soc. A **302**, 529 (1968).
[90] W. D. Hayes, Proc. Roy. Soc. A **320**, 187 (1970).
[91] R. L. Dewar, Phys. Fluids **13**, 2710 (1970).
[92] W. L. Jones, Rev. Geophys. Space Phys. **9**, 917 (1971).
[93] R. L. Dewar, J. Plasma Phys. **7**, 267 (1972).
[94] W. L. Jones, J. Fluid Mech. **58**, 737 (1973).
[95] R. L. Dewar, Phys. Fluids **16**, 1102 (1973).
[96] W. D. Hayes, Proc. R. Soc. Lond. A **332**, 199 (1973).
[97] D. B. Melrose, Austral. J. Phys. **34**, 563 (1981).
[98] H. Heintzmann and M. Novello, Phys. Rev. A **27**, 2671

(1983).
[99] Y. Takahashi, Fortschr. Phys. **34**, 323 (1986).
[100] G. W. Kentwell and D. A. Jones, Phys. Rep. **145**, 319 (1987).
[101] R. Kulsrud and A. Loeb, Phys. Rev. D **45**, 525 (1992).
[102] A. J. Brizard, D. R. Cook, and A. N. Kaufman, Phys. Rev. Lett. **70**, 521 (1993).
[103] A. J. Brizard and A. N. Kaufman, Phys. Rev. Lett. **74**, 4567 (1995).
[104] I. Y. Dodin and N. J. Fisch, Phys. Rev. Lett. **107**, 035005 (2011).
[105] I. Y. Dodin and N. J. Fisch, Phys. Plasmas **19**, 012102 (2012).
[106] I. Y. Dodin and N. J. Fisch, Phys. Plasmas **19**, 012103 (2012).
[107] I. Y. Dodin and N. J. Fisch, Phys. Plasmas **19**, 012104 (2012).
[108] I. Y. Dodin and N. J. Fisch, Phys. Plasmas **17**, 112118 (2010).
[109] S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity* (Wiley, New York, 1972).
[110] Regardless of how a is defined in a specific case [including whether a is a scalar or a vector, in which case the polarization is also inferred consistently from Eq. (11)], the resulting equations will be equivalent. Also, for linear waves, it is often convenient instead to have as independent variables the two complex-conjugate envelopes (Sec. VI), and this does not affect our analysis either.
[111] The independent functions are (a, ξ) but not (a, k_μ) because the latter, albeit describing the envelope, do not completely determine the wave field.
[112] The ACT also permits generalization to resonant interactions. For example, the beat phase $\Delta\xi \doteq \xi_1 - \xi_2$ of two resonant waves (denoted as 1 and 2) is a slow variable, so it may enter the GO Lagrangian density; i.e., $\mathfrak{L} = \mathfrak{L}(a_1, a_2, k_{1\mu}, k_{2\lambda}, \Delta\xi; x^\nu)$. Then, $\delta_{\xi_1} S = 0$ yields $\mathcal{J}_1{}^\mu{}_{;\mu} = \mathfrak{L}_{\Delta\xi}$, and $\delta_{\xi_2} S = 0$ yields $\mathcal{J}_2{}^\mu{}_{;\mu} = -\mathfrak{L}_{\Delta\xi}$. Hence the total action is conserved, $(\mathcal{J}_1^\mu + \mathcal{J}_2^\mu)_{;\mu} = 0$, which is a reincarnation of the Manley-Rowe theorem [67, 103].
[113] Notice that we define the EMT sign in a nonconventional way so the expressions be simplified for the adopted metric signature.
[114] For $g_{ij} = \operatorname{diag}(-1, 1, 1, 1)$ and a Lagrangian density $\mathcal{L}(\psi, \psi_{,\mu})$, the EMT [113], satisfying $T_\alpha{}^\beta{}_{,\beta} = 0$, is $T_\alpha{}^\beta \doteq -\psi_{,\alpha}\,\mathcal{L}_{\psi_{,\beta}} + \delta_\alpha^\beta \mathcal{L}$ [69, Sec. 32]. In particular, this explains why Eq. (14) is expected from Eq. (11), since $\mathfrak{L}_{a_{,\beta}} \equiv 0$.
[115] F. J. Belinfante, Physica **6**, 887 (1939).
[116] F. J. Belinfante, Physica **7**, 449 (1940).
[117] W. Pauli, Rev. Mod. Phys. **13**, 203 (1941).
[118] J. A. McLennan, Physica **32**, 689 (1966).
[119] J. D. Logan, *Invariant variational principles* (Academic Press, New York, 1977).
[120] J. E. Avron, E. Berg, D. Goldsmith, and A. Gordon, Eur. J. Phys. **20**, 153 (1999).
[121] Mind that $\mathcal{T}^{\alpha\beta}$ is the EMT of the wave rather than of the whole system, so it is often perceived solely as a mathematical abstraction and denied having a physical meaning [150]. However, the very existence of such a thing as the "elementary physical meaning" [150] is questionable in the general case, and, in a sense, the properties of the wave "physical" EMT render it even less real than the however abstract $\mathcal{T}^{\alpha\beta}$ (cf. Sec. V).
[122] G. S. Bisnovatyi-Kogan and O. Yu. Tsupko, Mon. Not. R. Astron. Soc. **404**, 1790 (2010).
[123] J.-Qi Shen, J. Opt. A: Pure Appl. Opt. **6**, 239 (2004).
[124] H. Goldstein, *Classical mechanics* (Addison-Wesley, Reading, MA, 1950).
[125] Then the particle canonical mass [Eq. (34)] is expected to be medium-dependent, as indeed found in solid state theory [151, Chap. 12] and also for ponderomotively-driven classical particles [139, 152, 153].
[126] L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinemann, Oxford, 1976).
[127] For electromagnetic waves in vacuum this agrees with the conventional definition through the Poynting vector (Sec. VI F).
[128] V. B. Berestetskii, E. M. Lifshits, and L. P. Pitaevskii, *Quantum electrodynamics* (Pergamon Press, New York, 1982), 2nd ed.
[129] The singularity of a circularly polarized field is also understood from the fact that its polarization ellipse has degenerate orientation. For details see Ref. [154] and papers on singular optics cited therein.
[130] J. H. Hannay, J. Phys. A Math. Gen. **18**, 221 (1985).
[131] M. V. Berry, Proc. Roy. Soc. Lond. A **392**, 45 (1984).
[132] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).
[133] L. Allen, M. J. Padgett, and M. Babiker, in *Progress of Optics, Vol. XXXIX* (Elsevier, New York, 1999), edited by E. Wolf; pp. 294-372.
[134] S. M. Barnett and L. Allen, Opt. Comm. **110**, 670 (1994).
[135] E. A. Startsev and R. C. Davidson, Nucl. Instr. Meth. Phys. Res. A **606**, 42 (2009).
[136] Particles are assumed elementary in the sense that their momenta are determined entirely by the translational velocities. This assumption is invalid for calculating relativistic corrections to $\Delta\boldsymbol{\rho}$ when particles are anisotropic. Indeed, an anisotropic medium experiences a torque from the wave when $\mathbf{v}_g$ is not parallel to $\mathbf{k}$ (Sec. IV C). As the angular momenta of particles change then, their relativistic centers of mass [155, 156] drift and thus carry an additional momentum not included in our model.
[137] See Eqs. (35) and (38) in Ref. [66].
[138] Notice that a medium consisting of *multiple* fluids generally would not have a uniquely defined rest frame.
[139] I. Y. Dodin and N. J. Fisch, Phys. Rev. E **77**, 036402 (2008).
[140] I. Y. Dodin, V. I. Geyko, and N. J. Fisch, Phys. Plasmas **16**, 112101 (2009).
[141] J. D. Jackson, *Classical electrodynamics* (Wiley, New York, 1975).
[142] T. H. Stix, *Waves in plasmas* (AIP, New York, 1992).
[143] F. N. H. Robinson, Phys. Rep. **16**, 313 (1975).
[144] A. Bers, in *Plasma Physics*, edited by C. DeWitt and J. Peyraud (Gordon and Breach, New York, 1975), Sec. II.
[145] J. R. Cary and A. N. Kaufman, Phys. Rev. A **21**, 1660 (1980).
[146] R. W. B. Best, Plasma Phys. Control. Fusion **27**, 691 (1985).
[147] A. L. Kholmetskii, O. V. Missevitch, and T. Yarman, Phys. Scr. **83**, 055406 (2011).
[148] I. Y. Dodin and N. J. Fisch, Phys. Rev. D **82**, 044044

(2010).
[149] I. Y. Dodin and N. J. Fisch, Phys. Plasmas **17**, 112113 (2010).
[150] M. E. McIntyre, J. Fluid Mech. **106**, 331 (1981).
[151] N. W. Ashcroft and N. D. Mermin, *Solid state physics* (Thomson Learning, Toronto, 1976).
[152] I. Y. Dodin and N. J. Fisch, AIP Proc. **1061**, 263 (2008) [in *Frontiers in Modern Plasma Physics: International Workshop on the Frontiers of Modern Plasma Physics, ICTP, Trieste, Italy, Jul 14-25* (AIP, New York, 2008)].
[153] A. I. Zhmoginov, I. Y. Dodin, and N. J. Fisch, Phys. Rev. E **81**, 036404 (2010).
[154] I. Freund, M. S. Soskin, and A. I. Mokhun, Opt. Comm. **208**, 223 (2002).
[155] V. V. Dixit and J. L. Pietenpol, Phys. Rev. A **26**, 53 (1982).
[156] C. Møller, *The theory of relativity* (Clarendon Press, Oxford, 1955), 3rd edn.

The Princeton Plasma Physics Laboratory is operated
by Princeton University under contract
with the U.S. Department of Energy.

Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2245
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov