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Toroidal Precession as a Geometric Phase

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Toroidal precession is commonly understood as the orbit-averaged toroidal drift of guiding centers in axisymmetric and quasisymmetric configurations. We give a new, more natural description of precession as a geometric phase effect. In particular, we show that the precession angle arises as the holonomy of a guiding center's poloidal trajectory relative to a principal connection. The fact that this description is physically appropriate is borne out with new, manifestly coordinate-independent expressions for the precession angle that apply to all types of orbits in tokamaks and quasisymmetric stellarators alike. We then describe how these expressions may be fruitfully employed in numerical calculations of precession.

I. INTRODUCTION

Toroidal precession can be described as the orbit-averaged toroidal drift[1] of guiding centers in axisymmetric or quasisymmetric configurations. It is a well-known[2,3] and well-studied[4,5] phenomenon that plays a fundamental role in a number of resonant processes in tokamaks and quasisymmetric stellarators. For instance, it can provide a mechanism for stabilizing the resistive wall mode and sawtooth oscillations, as in Refs. 6, 7, 8 and 9. Or it can lead to ripple diffusion in the presence of discrete coils[10].

The precession angle is typically calculated in one of two ways. One approach[3] is based on expressing the guiding center equations of motion in magnetic coordinates. The evolution equation for the toroidal angle is first expressed as a time integral over one bounce period, which is then converted to an integral over the poloidal angle. Another approach[4] is based on expressing the guiding center equations in action-angle coordinates. The action variables are first calculated by integrating the guiding-center Lagrange one-form around two fundamental cycles on each invariant torus. Then the Hamiltonian is expressed in terms of the actions so that the frequencies of motion can be calculated by differentiation. In general, existing methods for calculating the precession angle are closely tied to particular coordinate systems, be it flux coordinates or otherwise.

This is a curious state of affairs because the precession angle is a physically meaningful quantity[11] that is in no way dependent on the coordinate system chosen to descirbe dynamics. It means the existing interpretation of precession as orbit averaged drift, while technically correct, is somewhat unnatural. A natural interpretation would reflect the coordinate-independent nature of precession by providing coordinate-independent expressions for the precession angle.

The present work has emerged as a result of our desire to find such an interpretation. The latter developed as we studied problems where a tokamak's symmetry axis varies slowly with time. In these cases, it was more direct to work with the god-given cartesian coordinates than a time-dependent system of flux coordinates. This observation led us to search for alternative ways to express the precession angle that were not tied to special coordinate systems, a search that ultimately left us wanting a deeper geometric understanding of precession.

What has emerged from our investigation is both satisfying and surprising. Satisfying because we have indeed found a nice geometric interpretation of precession that leads to coordinate-independent expressions for the precession angle in arbitrary[12] quasisymmetric configurations. For tokamaks, these are our expressions 12 and 17, and for stellarators Eqs. 30 and 31. The surprise is the relationship between these expressions and the now-famous *geometric phases* studied in quantum mechanics[13], and elsewhere in plasma physics[14–16].

The canonical example of a geometric phase was discovered by Berry[17], who studied problems similar to the following. Suppose a spin 1/2 particle evolves in a uniform magnetic field whose direction is slowly cycled. After a single cycle, the quantum adiabatic theorem dictates that the particle's state, $|\psi|$, remains the same. However, the particle's *phase* acquires a contribution proportional to the solid angle subtended by the magnetic field vector in the course of the cycle. This is the Berry phase. It is known as a 'geometric' phase because it only depends on the geometry of the system's path in parameter space. It is also an example of *holonomy*, which is the defining feature of geometric phases in general. We have included an introduction to the mathematics behind holonomy in Appendix A for the interested reader.

It turns out that toroidal precession has *exactly the same structure* as Berry's phase; it is a physical manifestation of a geometric phase. This point is not obvious at first glance. But the seeds of the relationship are contained in a suggestive analogy. To wit, if one identifies a guiding center's "state" and "phase" with its location in the constants of motion space (E, μ, p_ϕ) and its toroidal angle, respectively, then the precession angle *begs* to be interpreted as a geometric phase. Indeed, after a guiding center's poloidal location is cycled in the course of a bounce period, its state (E, μ, p_ϕ) remains unchanged because (E, μ, p_ϕ) are constants of motion. However, its phase is altered by an amount equal to the precession angle. Cycling the guiding center's poloidal location is analogous to cycling the magnetic field direction in the spin 1/2 example above. Likewise, the conservation of the state variables is analogous to the preservation of $|\psi|$. The only way to complete this analogy, then, is to identify the precession angle with Berry's phase.

Of course, this suggestive analogy lacks in a few respects. In particular, it does not suggest what the analogue of the solid angle should be in the case of precession. That is, if Berry's phase is equal to the solid angle of the path in parameter space, then there should be some geometric property of curves in the poloidal plane that gives the precession angle, and the analogy does not indicate what this could be.

We will therefore proceed to *systematically* fill in all of the details of this analogy via a careful analysis of the guiding center equations of motion. Notably, it will become clear that the role played by solid angle in the spin 1/2 example must be filled by what we dub *precession holonomy*. This quantity, like solid angle, is an example of holonomy[18,19], meaning it fits the definition of a geometric phase. It can be ascribed to any curve in the 'parameter space', which in our case is the poloidal plane. Likewise, a poloidal trajectory's precession holonomy gives the precession angle, just as the solid angle of a path in parameter space gave the Berry phase. We will then show how to calculate the precession holonomy. By a stroke of luck, the resulting formula will turn out to be easy to express without reference to coordinates.

The paper is structured as follows. In the first part of the paper, sections II, III, and IV, we treat the special case of tokamaks, rather than more general quasisymmetric configurations. Section II contains a discussion of how ideas from the theory of geometric phases may be used to help reconstruct a guiding center's toroidal dynamics from its poloidal dynamics. To compliment this discussion, we have also included an introduction to geometric phases in Appendix A. This sets the stage for us to introduce the precession holonomy for tokamaks in section III. Section IV then completes our treatment of precession in tokamaks. There we derive a coordinate-independent expression for the tokamak precession holonomy and discuss how it can be used to numerically calculate the precession angle. After this, we leave the small world of tokamaks, and turn to arbitrary quasisymmetric configurations in section V. In this final section, we generalize the results from the first part of the paper to obtain coordinate independent expressions for the precession angle in any quasisymmetric stellarator.

II. CONNECTIONS AND RECONSTRUCTION

In this section, we will apply ideas from Ref. 19 related to *reconstruction* to guiding center motion in tokamaks. Reconstruction refers to the process of obtaining a full solution to a system of Hamiltonian equations with symmetry from a solution in the reduced phase space. As the authors show in Ref. 19, the general reconstruction procedure has an elegant formulation in terms of *principal connections*, which we will define in our context shortly. These connections, one of which we will introduce below, are intimately related to geometric phases, as we discuss in Appendix A. Ultimately, this insight will be what leads to our new expressions for the precession angle.

We start from the non-canonical Hamiltonian formulation of guiding center motion found by Littlejohn[20–22]. The guiding center phase space, P, is the cartesian product of the physical domain D and the parallel velocity axis, $P = D \times \mathbb{R}$. The symplectic structure on P is given by the Lagrange tensor $-d\vartheta$, where

$$\vartheta(x, v_\parallel) = q\mathbf{A}(x) \cdot dx + mv_\parallel b(x) \cdot dx, \tag{1}$$

$\mathbf{A}$ is the magnetic field's vector potential, q the particle charge, m the particle mass, and b the unit vector in the direction of the magnetic field. Note that we will not always use bold characters for vectors. This Lagrange tensor, together with the Hamiltonian,

$$H(x, v_\parallel) = \mu|B|(x) + \frac{1}{2}mv_\parallel^2, \tag{2}$$

determine the dynamical vector field $X : P \to TP$ via Hamilton's equations[23] $\mathrm{i}_X d\vartheta = -dH$. X captures all of the lowest order drifts, but contains small corrections that guarantee it defines a genuine Hamiltonian system. For example, this means it exactly conserves the energy given by Eq. 2 and the phase-space volume $d\vartheta \wedge d\vartheta$.

We will assume that $\mathbf{A}$ is chosen to be invariant under rotations of the physical domain D about the z-axis. Noether's theorem[23] then gives the conservation of canonical angular momentum. This invariant can be conveniently expressed in terms of the infinitesimal generator of rotations about the z-axis, $\xi(x) = Re_\phi$, where R is the major radius, using the formula

$$p_\phi = \vartheta(\xi) = q\mathbf{A} \cdot \xi + mv_\parallel b \cdot \xi. \tag{3}$$

We will also assume that the toroidal (azimuthal) component of $\mathbf{B}$ never vanishes. This allows for swapping the $v_\parallel$ coordinate with p_ϕ.

Because p_ϕ is a constant of motion, guiding center dynamics in the four-dimensional phase space takes place on the three-dimensional level sets $p_\phi = const$. Therefore, we can restrict attention to a single level set $p_\phi = l$. In the $(x, v_\parallel)$ coordinates, this level set has non-trivial shape. However, in (x, p_ϕ) coordinates, it is simply a copy of D. Thus, the dynamics on $p_\phi = l$ are given by a vector field $\dot{x}_l : D \to TD$ defined on D:

$$\dot{x}_l = -\frac{\nabla H \times b}{B_\parallel^*} + \frac{\partial H}{\partial p_\phi}\xi, \tag{4}$$

where $B_\parallel^* = q|B| + \lambda|B|b\cdot\nabla\times b$, $\lambda = (p_\phi - q\mathbf{A}\cdot\xi)/(\mathbf{B}\cdot\xi)$, and now H is expressed in (x, p_ϕ) coordinates as $H = \mu|B| + |B|^2\lambda^2/2m$. In particular, $\partial H/\partial p_\phi = |B|^2\lambda/(m\mathbf{B}\cdot\xi)$. We now shift our attention to analyzing the integral curves (field lines) of this vector field.

Because $\dot{x}_l$ is invariant under rotations about the z-axis, its poloidal component gives complete information about the poloidal dynamics, i.e.

$$(\dot{x}_l)_P = \dot{x}_l - \dot{x}_l \cdot e_\phi e_\phi = -\frac{\nabla H \times \xi}{\rho|\xi|^2}, \tag{5}$$

with $\rho = |B|B_\parallel^*/(\mathbf{B}\cdot\xi)$, gives a well defined dynamical system in the poloidal plane whose solutions are equal to the poloidal projection of the solutions to the full dynamical system defined by $\dot{x}_l$. Thus, the poloidal plane serves as the reduced phase space for guiding centers in tokamaks. In fact, it can be shown that $(\dot{x}_l)_P$ is a Hamiltonian system in this reduced phase space relative to a certain symplectic structure.

Conversely, the full trajectories of $\dot{x}_l$ can be "reconstructed" from the poloidal trajectories. In order to see how, it is extremely useful to take a short detour and introduce the notion of a principal connection on D[24]. This is merely an axisymmetric assignment to each point x in D a plane, $\mathbf{H}_x$, that is complementary to the span of $\xi(x)$. A typical $\mathbf{H}_x$ is depicted in Figure 1. Notice that if v_x is any vector emanating from the point x, then because $\mathbf{H}_x$ is complementary to the span of $\xi(x)$, v_x has a unique decomposition $v_x = \mathcal{A}(v_x)\xi(x) + \mathrm{hor}(v_x)$, where $\mathrm{hor}(v_x)$ is contained in $\mathbf{H}_x$ and $\mathcal{A}(v_x)$ is some number. Thus, hor and $\mathcal{A}$ are well-defined projection-like operations, with the former known as the horizontal projection and the latter the vertical projection. Because $\mathcal{A}$ acts linearly on vectors emanating from any fixed x in D, it is also known as a connection one-form. Note that, by the axisymmetry of the arrangement of planes $\mathbf{H}_x$, the connection one-form $\mathcal{A}$ must also be axisymmetric.

Returning to the issue of reconstructing full trajectories from poloidal trajectories, suppose some principal connection is prescribed on D. If we express $\dot{x}_l$ as

$$\dot{x}_l = (\dot{x}_l)_P + \dot{\phi}\xi \tag{6}$$

and apply the vertical projection operator, then we obtain

$$\dot{\phi} = \mathcal{A}\left[\dot{x}_l\right] - A\left[(\dot{x}_l)_P\right] \tag{7}$$

as an evolution equation for the toroidal angle. Here A is known as a *gauge field* associated to $\mathcal{A}$. In this case, it is a one-form whose action on vectors is given by first dropping the toroidal

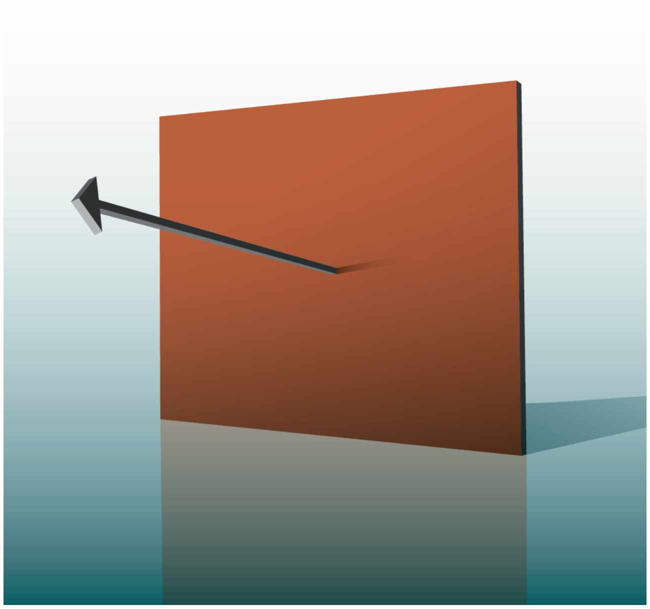

FIG. 1. A depiction of a typical $\mathbf{H}_x$. As is true for all such planes, the toroidal direction defined by $e_\phi(x)$, which in this case is depicted by an arrow, does not lie in $\mathbf{H}_x$. If the subspace spanned by $e_\phi(x)$ is denoted $\mathbf{V}_x$, then we have $T_xD = \mathbf{H}_x \oplus \mathbf{V}_x$.

component and then applying $\mathcal{A}$. Due to axisymmetry, both sides of Eq. 7 are functions independent of toroidal angle ϕ. Therefore the whole expression can be evaluated along a poloidal trajectory $\gamma(t)$ and integrated in time for one bounce period T. The resulting expression for the precession angle is then

$$\Delta\phi = \int_0^T \mathcal{A}\left[\dot{x}_l\right](\gamma(t))dt - \oint A. \tag{8}$$

The second term is the integral of the gauge field A around the loop traced out by the poloidal trajectory γ. It is known as the holonomy[19] of the connection around γ. Thus, the second term represents a geometric phase. Notice that it is manifestly independent of how the integration curve is parameterized. The first term does not have this nice property; in general, the time integral requires detailed information about the time dependence of the poloidal trajectory. For this reason, the latter is known as the dynamic phase.

If a different connection were to be used, then the geometric and dynamic phases would be different. However they would always sum to give the same result $\Delta\phi$. In particular, if a connection could be found such that $\dot{x}_l$ was horizontal, i.e. $\mathcal{A}(\dot{x}_l) = 0$, then $\Delta\phi$ would be realized as a purely geometric phase. In the next section we will present such a connection.

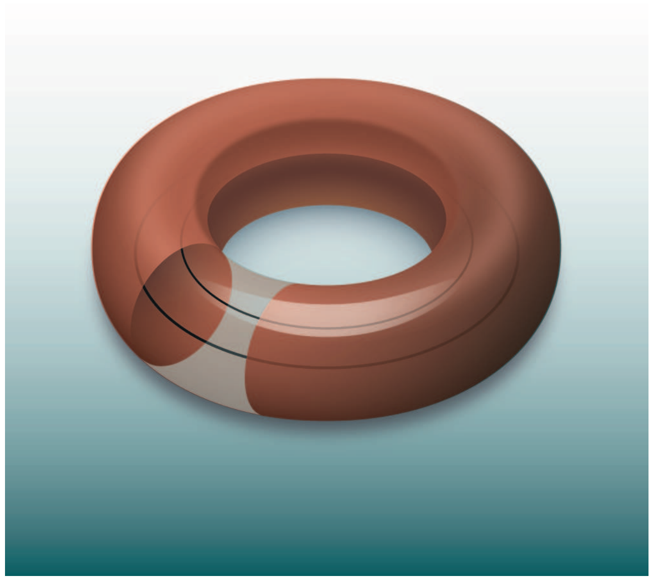

FIG. 2. An example toroidal region, D, with the FPSOs displayed as toroidal rings. Typically, for fixed values of μ and p_ϕ, there will be two or three FPSOs, one corresponding to the center of the passing trajectories, another to the center of the trapped trajectories, and a third to the tip of the largest banana orbit. In the case depicted, perhaps the tip of the largest banana tip lies outside of the toroidal domain, leaving just two FPSOs within D.

III. A CONNECTION FOR THE TOROIDAL PRECESSION

If a connection renders $\dot{x}_l$ horizontal, then for each x in D, $\mathbf{H}_x$ must contain $\dot{x}_l$ whilst remaining complimentary to the span on ξ. Such a plane will certainly not exist at those points where $\dot{x}_l$ is parallel to ξ. These points correspond to zeros of the poloidal vector field $(\dot{x}_l)_P$. Moreover, by axisymmetry, if x is one such point, then the entire circle of revolution passing through x must consist of similar points (see Figure 2). These circles of revolution are known as the fixed-point stagnation orbits[3,25], or FPSOs. Due to the presence of these FPSOs, a connection that renders $\dot{x}_l$ horizontal must have a restricted domain of definition that *at least* excludes the FPSOs, if not even more points. Therefore, in what follows we will assume that D has had its FPSOs removed.

On this redefined D, it turns out there is a connection that at once renders $\dot{x}_l$ horizontal and exists at all points. This means that it is not necessary to remove any more points from D than we already have. For each point x in D, this connection's horizontal subspace $\mathbf{H}_x$

is equal to the span of the vectors $\mathbf{D}H(x)$ and $\dot{x}_l(x)$, where

$$\mathbf{D}H = \nabla H - \frac{b \cdot \nabla H}{b \cdot \xi}\xi. \tag{9}$$

The following simple reasoning will show

$$\dot{x}_l \cdot \mathbf{D}H \times \xi = \dot{x}_l \cdot \nabla H \times \xi \neq 0, \tag{10}$$

and that $\mathbf{H}_x$ is an axisymmetric assignment of planes. This will prove that $\mathbf{H}_x$ gives a well defined principal connection.

First of all, note that none of the three vectors that appear in Eq. 10 vanish on the region we are considering; ξ does not have zeros, if $\dot{x}_l$ were zero, then $(\dot{x}_l)_P$ would be zero, and ∇H only vanishes when $(\dot{x}_l)_P$ does. Then notice that $\dot{x}_l$ and ξ must be tangent to the surfaces defined by $H = const.$ (remember that $p_\phi = l$ now). This much proves Eq. 10. Finally, because $\mathbf{D}H$ and $\dot{x}_l$ are axisymmetric vectors, the planes $\mathbf{H}_x$ are manifestly arranged in an axisymmetric manner.

By identifying this connection, in accordance Eq. 8, the toroidal precession angle is now seen to be a purely geometric phase. This can be checked directly by evaluating the dynamic phase in Eq. 8 using an expression for the connection one-form $\mathcal{A}$. The latter is simply given by

$$\mathcal{A} = \frac{\dot{x}_l \times \mathbf{D}H \cdot dx}{\dot{x}_l \times \mathbf{D}H \cdot \xi}. \tag{11}$$

We will derive an expression for the holonomy of this connection, which we call *precession holonomy*, in the next section.

IV. PRECESSION HOLONOMY AND ITS INTERPRETATION

In order to calculate the precession holonomy of poloidal trajectories in tokamaks, all we must do is substitute the expression for the connection one-form given in the previous section into Eq. 8. Because $\mathcal{A}(\dot{x}_l) = 0$, the dynamic phase is 0, leaving just the geometric phase to calculate. The latter is given by

$$\begin{aligned} \Delta\phi &= -\oint_C A = -\oint_C \frac{\dot{x}_l \times \mathbf{D}H \cdot dx}{\dot{x}_l \times \mathbf{D}H \cdot \xi} \\ &= -\oint_C \frac{\mathbf{B} \cdot dx}{\mathbf{B} \cdot \xi} - \oint_C \frac{\partial H/\partial p_\phi}{(\nabla H)^2} \nabla H \times \mathbf{B}^* \cdot dx, \end{aligned} \tag{12}$$

where $\mathbf{B}^* = \nabla \times (q\mathbf{A} + \lambda \mathbf{B}) = \rho\xi$. Recall that, at this point, C is the closed curve in the poloidal plane defined by the equations $p_\phi = l$, $H = E_o$, and $\phi = \phi_o$, where l and E_o are a particle's canonical angular momentum and energy, respectively, and ϕ_o is an arbitrary toroidal angle. But as we will show when we generalize to quasisymmetric configurations, the above expression will remain valid even when C is continuously deformed within the drift surface. Thus, Eq. 12 is a very flexible, coordinate-independent expression for the precession angle, $\Delta\phi$.

Eq. 12 can also be expressed in terms of surface integrals using Stoke's theorem. But before naively converting the line integral in Eq. 12 into a surface integral, we must account for the fact that D does not contain the FPSOs. The presence of these 'holes' in D implies that C need not be the boundary of some closed two-dimensional surface contained in D. However, if we add to C tiny loops, L_i, such that the i'th tiny loop encircles the i'th FPSO encircled by C, then the resulting curve *is* the boundary of some two-dimensional surface S. In symbols, $C' \equiv C + \sum_i L_i = \partial S$. C' is also depicted in Figure 3.

Stoke's theorem can now be applied in the obvious way to convert the line integral around C' into a surface integral over S. We choose the orientations of the L_i to be opposite to that of C so that the result is

$$\oint_C A = -\sum_i \oint_{L_i} A + \int_S dA, \tag{13}$$

where the integrand dA is known as the *curvature* of the connection. It can be calculated as follows. If $A = \mathbf{w} \cdot dx$, for some vector field $\mathbf{w}$, then dA is simply $\nabla \times \mathbf{w} \cdot dS$.

Typically in tokamaks, a guiding center will encircle just one FPSO, so $i = 1$ is the only term in the sum. Moreover, this FPSO usually corresponds to an elliptic fixed point of $(\dot{x}_l)_P$. Thus, if L_1 is chosen to lie on one of the constant energy tori who stay very close to the FPSO, then $\oint_{L_1} \mathcal{A} \approx T_o\dot{\phi}_o$, where T_o is the deeply trapped (or deeply passing) bounce (or circulation) period, and $\dot{\phi}_o$ is $\dot{\phi}$ evaluated on the FPSO. With L_1 chosen in this way, Eq. 13 can be re-written

$$\oint_C A = -T_o\dot{\phi}_o + P\int_{S_o} dA, \tag{14}$$

where S_o is a surface whose boundary consists of C and a single point on the FPSO. The P in front of the surface integral denotes a principal value evaluation. It should be calculated as follows. Let E_ϵ be a continuous family of energy values such E_o is the energy of the

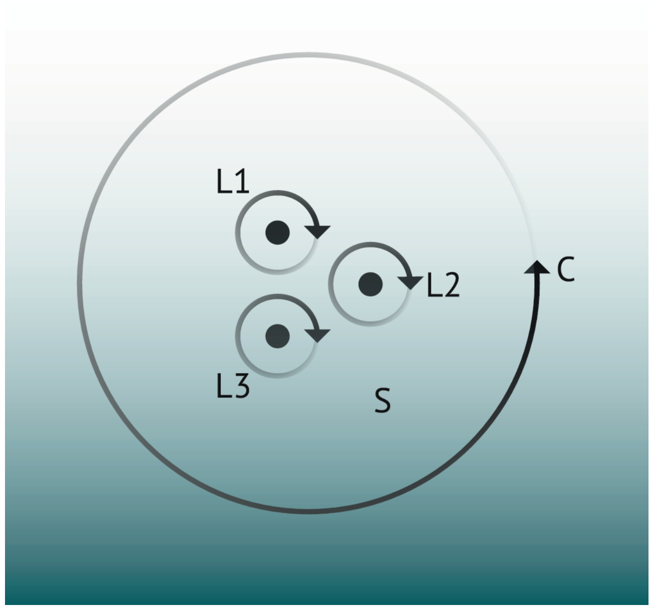

FIG. 3. A schematic of the curve in the poloidal plane C'. The precession angle is given by integrating the gauge field around C. In order to convert the latter line integral into a surface integral, the gauge field's singularities must be carefully accounted for. These singularities, denoted here with black dots, occur on the FPSOs. By encircling them with smaller loops L_i and appending these L_i to the original integration path, Stoke's theorem may be applied to convert the line integral of the gauge field over the modified integration path into a surface integral over S, the region bounded by the modified integration path. This then gives the precession angle as a surface integral plus residues from the FPSOs.

stagnation orbit. Let S_ϵ be a family of surfaces such that for every $\epsilon > 0$, ∂S_ϵ consists of C and a loop L_ϵ that tightly encircles the stagnation orbit while staying in the surface $H = E_\epsilon$. Then

$$P \int_{S_o} dA = \lim_{\epsilon \to 0} \int_{S_\epsilon} dA. \tag{15}$$

Now we bring Eq. 14 into its final form by making use of the identity

$$\begin{aligned} dA = & -\frac{1}{\rho} \nabla \cdot \left(\rho \frac{\partial H}{\partial p_\phi} \frac{\nabla H}{(\nabla H)^2} \right) \mathbf{B}^* \cdot dS \\ & + \frac{1}{(b \cdot \xi)^2} \Big(b \cdot \nabla \times b \Big) \xi \cdot dS. \end{aligned} \tag{16}$$

Substituting this into Eq. 14, we obtain the precession angle in terms of surface integrals and a residue from the FPSOs

$$\begin{aligned}\Delta\phi =& T_o\dot{\phi}_o + P\int_{S_o}\frac{1}{\rho}\nabla\cdot\left(\rho\frac{\partial H}{\partial p_\phi}\frac{\nabla H}{(\nabla H)^2}\right)\mathbf{B}^*\cdot dS \\ &-\int_{S_o}\frac{1}{(b\cdot\xi)^2}\Big(b\cdot\nabla\times b\Big)\xi\cdot dS.\end{aligned} \tag{17}$$

The first term represents the precession angle of the FPSO, as it is the limiting value the precession angle assumes in the deeply trapped (or passing) limit. The second term is a measure of the amount of "charge", or source, for the field $\mathbf{F} = \frac{\partial H}{\partial p_\phi}\frac{\nabla H}{(\nabla H)^2}$ contained in a constant energy torus. The appropriate way to calculate this charge is by taking the divergence of $\mathbf{F}$ relative to the volume form $\sigma = \rho dx\wedge dy\wedge dz$. σ is actually the volume form induced by the Liouville volume, $d\vartheta\wedge d\vartheta$, so we have dubbed this term a symplectic charge. Finally, the last term measures the total amount of magnetic torsion $b\cdot\nabla\times b$ contained in the same constant energy torus. This term does not have to be evaluated as a principal value because the integrand is non-singular. Moreover, it gives the holonomy of *another* principal connection defined on D. For this connection, the horizontal subspace at a point x in D is defined to be those vectors that are perpendicular to $b(x)$. Its vertical projection is given by

$$\mathcal{A}_b = \frac{b\cdot dx}{b\cdot\xi}. \tag{18}$$

Notice that this connection appears in the definition of $\mathbf{D}H$. Indeed, $\mathbf{D}H = \nabla H - \mathcal{A}_b(\nabla H)\xi = \mathrm{hor}_b(\nabla H)$, the horizontal part of the gradient.

Between equations 12 and 17, we now have a pair of complimentary coordinate-independent expressions for the precession angle in an arbitrary tokamak, the first giving a line integral representation and the second a surface integral representation. Because these expressions realize the precession angle as the precession holonomy of a poloidal trajectory, they also demonstrate that precession, like Berry's phase, is a type of geometric phase phenomenon. Therefore we can see that it is physically appropriate to interpret toroidal precession as a geometric phase.

Aside from these expressions' conceptual appeal, they are also useful for practical numerical calculations of the precession angle. This is because all of the quantities that appear in these expressions can be computed in a straightforward fashion in any coordinate system. In particular, in up-down symmetric configurations, Eq. 12 easily leads to a way to evaluate the precession angle in $(|B|, \psi_p)$-coordinates, where $\psi_p = \mathbf{A} \cdot \xi$. In these coordinates, the integration curve in Eq. 12 can be parameterized by ψ_p explicitly using energy conservation,

$$|B|(\psi_p) = \frac{-m\mu + \sqrt{m^2\mu^2 + 2Hm\lambda^2(p_\phi, \psi_p)}}{\lambda^2(p_\phi, \psi_p)}, \tag{19}$$

where H is the relevant particle's energy, and $\lambda = (p_\phi - q\psi_p)/(\mathbf{B} \cdot \xi)$. Once the components of the metric tensor and the major radius R are found in these coordinates, this parameterization can be used to reduce the calculation of the precession angle to one ψ_p integral for each relevant value of (E, μ, p_ϕ). Simple root finding is necessary to determine the limits of integration for these integrals. This entire procedure poses no serious difficulty because $|B|$, ψ_p, and R can readily be extracted from existing equilibrium reconstruction codes.

Note that because no approximations are made on the integration path, this approach easily accounts for all finite-orbit-width effects, and works just as well for passing particles as for trapped particles. Also note that all of the functions that appear in our integrals that cannot be expressed analytically in terms of ψ_p and $|B|$ are functions of position only, i.e. they can be determined numerically before specifying a particles energy, angular momentum, or magnetic moment. These two features may make this approach an especially attractive one to implement in stability codes such as MISK[26] or MARS-K[27]. In these codes, functions of the precession frequency are integrated over the constants of motion space. In order to speed the evaluation of these integrals, thin banana approximations are often made, meaning finite-orbit-width effects are lost. The approach we have just outlined may allow these codes to retain finite-orbit-with effects while only incurring a minimal penalty in computation time.

V. GENERALIZATION TO QUASISYMMETRIC CONFIGURATIONS

Now we turn to the task of generalizing the results of the previous section to the more general class of quasisymmetric stellarators. Because these configurations are more complicated geometrically than those we have considered so far, our treatment of precession also acquires some complications. In particular, in this section we will draw much more heavily on Cartan's calculus of differential forms[23]. But, as we will show, while our treatment becomes more advanced, the expressions for the precession angle that we derive are only marginally more complicated than what we found for tokamaks. Readers not as interested in the analysis as the resulting expressions can skip straight to Eqs. 30 and 31, which we have expressed using standard vector calculus notation to the extent possible.

We begin with a convenient definition of a quasisymmetric magnetic field. A quasisymmetric magnetic field is a divergence free field $\mathbf{B}$ such that there exists a continuous family of spatial transformations $\Phi_\theta : D \to D$ parameterized by the angle θ that satisfies the following properties.

$$\begin{aligned} \Phi_{\theta_1} \circ \Phi_{\theta_2} &= \Phi_{\theta_1+\theta_2} \\ \Phi_0 &= \text{identity} \\ \Phi_\theta^* B &= B \\ \Phi_\theta^* (\mathbf{B} \cdot dx) &= \mathbf{B} \cdot dx \\ \Phi_\theta^* |B| &= |B|, \end{aligned} \tag{20}$$

where $B = \mathbf{i}_{\mathbf{B}}(dx \wedge dy \wedge dz)$ is the magnetic field expressed as a two-form.

This definition is only slightly different from the typical definition[28] of quasisymmetry. The condition $\Phi_\theta^* B = B$ is equivalent to the common requirement that the magnetic field possess flux surfaces. Likewise, the condition $\Phi_\theta^* |B| = |B|$ is very nearly the usual requirement that $|B|$ only depend on a single combination of Boozer angles. On the other hand, the condition $\Phi_\theta^* (\mathbf{B} \cdot dx) = \mathbf{B} \cdot dx$ is apparently absent from the usual definition. Nonetheless, this is an entirely sane condition to include in the definition of quasisymmetry for two reasons. First, by including this condition in the definition of a quasisymmetric magnetic field, it can be shown that quasisymmetric fields are *precisely* those magnetic fields that lead to guiding center dynamics that possess a spatial symmetry, and therefore a Noether conserved quantity, p_ϕ. Second, it may be shown that if a magnetic field with flux surfaces and symmetric $|B|$ is also an ideal equilibrium, $\mathbf{J} \times \mathbf{B} = \nabla p$, then $\Phi_\theta^* (\mathbf{B} \cdot dx) = \mathbf{B} \cdot dx$ is satisfied automatically. Thus, if we restrict attention to ideal equilibria, then the definition of quasisymmetry given here is completely equivalent to the usual one. Otherwise, while this definition may indeed be different from the usual one, it includes the absolute minimum set of assumptions on $\mathbf{B}$ to guarantee the existence of p_ϕ.

This Noether conserved quantity is an especially important ingredient in our generalization of the results from the first part of the paper. It is given by

$$p_\phi = \vartheta(\xi) = q\mathbf{A} \cdot \xi + m v_\parallel b \cdot \xi, \tag{21}$$

where the vector potential is chosen to be invariant under the family of transformations Φ_θ, and the vector ξ is given by

$$\xi(x) = \frac{d}{d\theta}\bigg|_{\theta=0} \Phi_\theta(x). \tag{22}$$

Note that it is formally identical to the invariant in the axisymmetric case. This formal similarity is the key to our generalization.

Indeed, just as in the axisymmetric case, we swap the $v_\parallel$ coordinate for p_ϕ and then restrict to the level set $p_\phi = l$. The equations of motion on this level set, which again is just a copy of D, then have the same form as in the axisymmetric case,

$$\dot{x}_l = -\frac{\nabla H \times b}{B_\parallel^*} + \frac{\partial H}{\partial p_\phi}\xi. \tag{23}$$

And we are then faced with formally the same problem in formulating precession as a geometric phase: we must find a connection that renders $\dot{x}_l$ horizontal. Likewise, the connection that accomplishes this task is formally the same: it is defined by the span of the vectors $\dot{x}_l$ and $\mathbf{D}H$.

However, there is a technical difference. The assignment of planes $\mathbf{H}_x$ in the axisymmetric case was required to axisymmetric. Because now axisymmetry has been replaced with quasisymmetry, the assignment of planes in the current setting should be quasisymmetric. In terms of the connection one-form, this means

$$\Phi_\theta^* \mathcal{A} = \mathcal{A}. \tag{24}$$

If $\mathbf{D}H$ were defined exactly as it was in the axisymmetric case, then the assignment of planes given by the span of $\dot{x}_l$ and $\mathbf{D}H$ would *not* be quasisymmetric, in general. The reason for this boils down to the fact that the ordinary gradient operator ∇ does not always commute with the transformations Φ_θ. The most unfortunate consequence of this failure to commute is that the ordinary gradient of a function that is invariant under the family of transformations Φ_θ need not be an invariant vector field. On the other hand, it is simple to define a new gradient operator that *does* commute with the transformations Φ_θ. First we define the *averaged metric tensor*, $\langle g \rangle$,

$$\langle g \rangle (u_x, v_x) = \frac{1}{2\pi} \int_0^{2\pi} (\Phi_\theta^* g)(u_x, v_x) d\theta, \tag{25}$$

where u_x, v_x are vectors emanating from an arbitrary point x in D, and g is the standard metric tensor on D. Then we define the *symmetric gradient* of a function f, $\nabla_{\mathrm{sym}} f$, in the same manner the usual gradient is defined, but with the usual metric tensor replaced by the averaged metric tensor:

$$\langle g \rangle (\nabla_{\mathrm{sym}} f, w) = df(w), \tag{26}$$

where w is an arbitrary vector field on D. By using the symmetric gradient of the Hamiltonian in the definition of $\mathbf{D}H$,

$$\mathbf{D}H = \nabla_{\mathrm{sym}} H - \mathcal{A}_b(\nabla_{\mathrm{sym}} H)\xi, \tag{27}$$

our definition of the connection can be shown to satisfy Eq. 24. In particular,

$$\mathcal{A} = \frac{\dot{x}_l \times \mathbf{D}H \cdot dx}{\dot{x}_l \times \mathbf{D}H \cdot \xi} \tag{28}$$

is a legitimate connection one-form of a principal connection that renders $\dot{x}_l$ horizontal in quasisymmetric stellarators.

Now we turn to calculating the precession holonomy of a poloidal trajectory. This holonomy will give the precession angle, $\Delta\Theta$. But before we present the calculation, we should be clear about what the precession angle is in a quasisymmetric configuration. After all, we cannot employ the simple toroidal coordinate ϕ that was available in the axisymmetric case. Indeed, we have not introduced any coordinates at all at this point. Therefore, the precession angle needs a coordinate independent definition, which is as follows. Suppose a particle completes a bounce or circulation period. Its position at the beginning of this period, a, is related to its position at the end of the period, b, via $\Phi_{\Delta\Theta}(a) = b$. $\Delta\Theta$ is the precession angle.

In order to calculate $\Delta\Theta$, we could, in principle, work out an expression for a gauge field A and then insert this into Eq. 8. However, this would require identifying a poloidal plane. In quasi-symmetric stellarators, there is not a single canonical choice of a poloidal plane - there are many such choices! Instead we will use a simple trick to calculate the holonomy that only makes use of the surfaces $H = \mathit{const.}$ and $\mathcal{A}$.

First, we recall the so-called Cartan structure equation[29],

$$d\mathcal{A}(\mathrm{hor}(v), \mathrm{hor}(w)) = d\mathcal{A}(v, w), \tag{29}$$

for any vectors v, w. This identity is valid for any principal connection. Because $\dot{x}_l$ and ξ form a basis for the planes tangent to a level set $H = const.$ and $\text{hor}(\xi) = 0$, the structure equation implies that $d\mathcal{A}$ vanishes when restricted to $H = const.$. Thus, $\mathcal{A}$ restricted to this level set is *closed*, meaning the integral of $\mathcal{A}$ along any curve in this level set will be left unchanged if the curve is continuously deformed within the level set, keeping the end points fixed. This elementary result is proven in any introductory text on differential geometry, for instance see Ref. 30.

Now we integrate $\mathcal{A}$ along a special curve contained in the surface $H = E_o$, which we will assume is compact and without boundary. The first arc of the curve, C_1, will follow some arbitrary initial condition a in $H = E_o$ along the trajectory of $\dot{x}_l$ passing through that point for one bounce period. Note that, by the classification of surfaces, the connected components of $H = E_o$ must be tori. Also note that the end points a and b of C_1 satisfy $b = \Phi_{\Delta\Theta}(a)$, by definition of the precession angle. The second arc of the curve, C_2, will then travel from the termination point of C_1, b, back to a along the field line of ξ that joins a with b. The total circuit $C = C_1 + C_2$ is then a closed loop that encircles the the torus containing a, T_a, one time poloidally. Therefore we may as well have integrated $\mathcal{A}$ along any other closed loop that encircles T_a once poloidally. However, our particular choice of curve makes the integral simple to evaluate. Indeed, because $\dot{x}_l$ is horiztonal, $\int_{C_1} \mathcal{A} = 0$. Moreover, because $\mathcal{A}(\xi) = 1$, $\int_{C_2} \mathcal{A} = -\Delta\Theta$. Therefore, we have derived the following coordinate-independent expression for $\Delta\Theta$:

$$\begin{aligned}\Delta\Theta &= -\oint_C \mathcal{A} = -\oint_C \frac{\dot{x}_l \times \mathbf{D}H \cdot dx}{\dot{x}_l \times \mathbf{D}H \cdot \xi} \\ &= -\oint_C \frac{\mathbf{B} \cdot dx}{\mathbf{B} \cdot \xi} - \oint_C \frac{\partial H/\partial p_\phi}{(\nabla_{\text{sym}} H)^2} \nabla_{\text{sym}} H \times \mathbf{B}^* \cdot dx,\end{aligned} \tag{30}$$

where C is *any* closed curve encircling T_a in the same sense as a trajectory of $\dot{x}_l$. Also, $(\nabla_{\text{sym}} H)^2 = \langle g \rangle (\nabla_{\text{sym}} H, \nabla_{\text{sym}} H)$. Applying Stoke's theorem in a similar manner as in the axisymmetric case, we also obtain

$$\begin{aligned}\Delta\Theta =& T_o \dot{\phi}_o + P \int_{S_o} \frac{1}{\rho} \nabla \cdot \left(\rho \frac{\partial H}{\partial p_\phi} \frac{\nabla_{\text{sym}} H}{(\nabla_{\text{sym}} H)^2} \right) \mathbf{B}^* \cdot dS \\ &- \int_{S_o} \frac{1}{(b \cdot \xi)^2} \Big(b \cdot \nabla \times b \Big) \xi \cdot dS.\end{aligned} \tag{31}$$

With Eqs. 30 and 31 in hand, we have now completed the generalization of the axisymmetric results to quasisymmetric configurations.

VI. CONCLUSION AND FUTURE WORK

We have thus succeeded in demonstrating that the precession angle in both axisymmetric and quasisymmetric configurations is a geometric phase. Moreover, we have shown that this physically appropriate interpretation leads to the coordinate-independent expressions for the precession angle, Eqs. 12 and 17 in tokamaks, and Eqs. 30 and 31 in quasisymmetric stellarators. Such coordinate independent expressions for the precession angle have been missing from the literature.

We have also used these expressions to identify a new method for calculating the precession angle numerically. The latter amounts to evaluating our Eq. 12 in $(|B|, \psi_p)$-coordinates. This method may be an attractive one to implement in stability codes such as MISK or MARS-K as it could allow for these codes to account for missing finite-orbit-width effects at only a minimal computational cost.

Our results are not applicable to fields with nulls in the toroidal field. These fields are of interest in the magnetic fusion community as they include those found in dipole experiments, RFPs, and FRCs. Therefore, it would be interesting to extend our results by allowing for toroidal field nulls.

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Appendix A: Elements of the theory of Geometric Phases

What is a geometric phase? Probably the most accessible example comes from spherical geometry. Suppose an unlucky ant is made to walk around a special circuit on the surface of a model globe, while performing an equally special stunt. The circuit begins at an arbitrary point A on the equator. It then proceeds north along a line of longitude until reaching the north pole. At the pole, the direction of the path abruptly rotates by θ radians before heading south again along the line of longitude defined by the path's new heading. Upon reaching the equator once more, the path finally returns to the starting point A along the equator. The stunt the ant is to perform while taking this journey consists of carrying a tiny rod at a constant angle relative to the direction it walks. When the ant changes direction at the 'kinks' in the path, it is not to rotate the rod under any circumstances.

If the ant manages to pull off this strange maneuver without a hitch, a keen observer would then notice that the initial and final orientations of the rod were different; the device would have suffered a rotation by θ radians. This rotation angle is a geometric phase or *holonomy*. It manifested itself through the ants endeavor to hold the rod parallel to itself while walking around the circuit.

More generally, geometric phases measure the rotation of 'geometric data' after it is transported 'parallel to itself' around a curve in a 'parameter space'. In the ant example, the geometric data were the vectors tangent to the sphere, while the parameter space was the sphere itself. Meanwhile, in the spin 1/2 example, the geometric data were the wavefunction phase factors, and the parameter space was the sphere representing different orientations of the magnetic field.

Therefore, in order to think of the precession angle as a geometric phase, we would like to regard toroidal phases as the geometric data to be transported, and the poloidal plane as parameter space. This much is clear in light of the analogy between Berry's phase and precession we are looking for. But what could it mean to "transport a toroidal phase parallel to itself"? In the ant example, parallel transport of a vector was intuitive; by keeping the angle between the rod and the path constant, the rod is always translated parallel to itself[31]. On the other hand, in the spin 1/2 example, and in precession especially, there does not appear to be an intuitive notion of parallelism.

A deeper look at the ant example offers clues. The key is formulating the ant's transport process in what we will call the space of geometric data, D. The latter is defined as the collection of all vectors tangent to the sphere, *together with their basepoints*, i.e. the sphere's tangent bundle[23]. We denote a typical element of this space by v_p, where p is a point on the sphere, the basepoint, and v is a vector that lies in the plane tangent to the sphere at p. We also define a projection map π which takes an arbitrary point in the space of geometric data, v_p, and returns the point in parameter space it is attached to, $\pi(v_p) = p$. Note that there are four independent directions one can travel in the space D. Two of these correspond to keeping the base point, p, fixed while varying the tangent vector v. These are known as the *vertical* directions. The other two correspond to the freedom in changing the basepoint p.

Suppose we prescribe an arbitrary circuit for the ant to walk. The circuit can be formalized as a curve $\gamma : [0,1] \to S^2$ on the sphere with $\gamma(0) = \gamma(1) = A$, where A is the circuit's starting point. If the ant transports an initial vector v_A 'parallel to itself' as it navigates the circuit, then it traces out a curve in D, $\tilde{\gamma}(t) = v(t)_{\gamma(t)}$, that satisfies $\pi(\tilde{\gamma}(t)) = \gamma(t)$. We say that ant has taken a curve in the parameter space, $\gamma(t)$, and produced *lifted* curve $\tilde{\gamma}(t)$ in the space of geometric data, D.

This lifted curve $\tilde{\gamma}$, like all curves, has a tangent vector $\tilde{\gamma}'(t)$ for each t. In particular, $u_{v_A} \equiv \tilde{\gamma}'(0)$ is a tangent vector at v_A, i.e. $u_{v_A} \in T_{v_A}D$, because $\tilde{\gamma}(0) = v_A$. If the circuit the ant walked were different, but the initial vector v_A was the same, then u_{v_A} would, in general, be a different vector in $T_{v_A}D$. However, the collection of u_{v_A} obtained by considering *all* of the possible circuits the ant could walk forms a linear subspace $\mathbf{H}_{v_A} \subset T_{v_A}D$ whose dimension is equal to the dimension of the parameter space. $\mathbf{H}_x$ is essentially a copy of T_AS^2. Moreover, no vector in $\mathbf{H}_{v_A}$ other than the zero vector points purely in one of the vertical directions in D. If there were such a vector in $\mathbf{H}_{v_A}$, it would correspond to a circuit on the sphere consisting of just one point. But on such a circuit, the ant will not alter the initial vector v_A, meaning the corresponding lifted curve in D would have zero velocity. Therefore, if $\mathbf{V}_{v_A} \subset T_{v_A}D$ denotes the span of vectors that point in the vertical directions, then $T_{v_A}D = \mathbf{H}_{v_A} \oplus \mathbf{V}_{v_A}$, where $\oplus$ denotes direct sum of vector spaces. This means that an arbitrary vector $w_{v_A} \in T_{v_A}D$ has a unique decomposition, $w_{v_A} = h_{v_A} + e_{v_A}$, with $h_{v_A} \in \mathbf{H}_{v_A}$ and $e_{v_A} \in \mathbf{V}_{v_A}$.

Because v_A, the initial vector for the transport process, is arbitrary, we come to the conclusion that the ant's rule for transporting vectors defines a two-dimensional subspace $\mathbf{H}_{v_p} \subset T_{v_p}D$ for each $v_p \in D$. Such an assignment of subspaces, which are called *horizontal subspaces*, is known as an *Ehresmann connection*[19] on D. An Ehresmann connection *splits* all of the tangent spaces to D into horiztonal and vertical subspaces, $T_{v_A}D = \mathbf{H}_{v_A} \oplus \mathbf{V}_{v_A}$.

Now suppose that, instead of specifying the ant's rule for transporting vectors, we only told the ant about the Ehresmann connection on D that we just discussed. Does this object contain enough information for the ant to deduce what the transport rule is? It turns out the answer is yes. Given an initial vector, if the ant endeavors to transport this vector so it traces out a curve $\tilde{\gamma}$ in D with $\tilde{\gamma}'(t) \in \mathbf{H}_{\tilde{\gamma}(t)}$, then it can be shown that the curve $\tilde{\gamma}$ it finds is *unique* and equal to the curve it would have produced had we given the ant the transport rule explicitly. Therefore, we see that the ant's rule for transporting vectors parallel to themselves can be losslessly encoded into an Ehresmann connection on D.

More generally, on an arbitrary space of geometric data, D, it is possible to specify an Ehresmann connection. Given a point $d \in D$, an Ehresmann connection is an assignment of a plane $\mathbf{H}_d$ such that $T_d D = \mathbf{H}_d \oplus \mathbf{V}_d$. We also have the following general theorem proven in Ref. 19.

Theorem 1 *Suppose an Ehresmann connection is specified on a space of geometric data D. If γ is a curve in the parameter space, and $\tilde{\gamma}$ is a lift of γ with fixed initial condition, then if $\tilde{\gamma}'$ is horizontal, $\tilde{\gamma}$ is unique.*

These two facts are used to *define* what it means to transport geometric data parallel to itself in general. Given a space of geometric data D, we first define an Ehresmann connection. Note that there is a good amount of freedom in this step. Then we suppose we are given a loop in the parameter space γ and an initial piece of geometric data d_o. The *parallel transport* of d_o along the curve γ is then defined as the unique lift of γ, $\tilde{\gamma}$, with the properties $\tilde{\gamma}(0) = d_o$ and $\tilde{\gamma}'(t) \in \mathbf{H}_{\tilde{\gamma}(t)}$.

Likewise, *holonomy* can be defined in this general context. First set P equal to the parameter space for our space of geometric data D. For an arbitrary point $p \in P$, set D_p equal to the collection of geometric data attached to the point p. Given a loop in P, $\gamma : [0,1] \to P$, that starts and ends at $p_o \in P$, an arbitrary $d \in D_{p_o}$ can be parallel transported along γ, giving the curve $\tilde{\gamma} : [0,1] \to D$. In particular, $\tilde{\gamma}(1)$ defines a new element of D_{p_o}, $\mathrm{hol}_{p_o}(d) \in D_{p_o}$. It is not difficult to show that the mapping $\mathrm{hol}_{p_o} : D_{p_o} \to D_{p_o}$, which implicitly depends on the loop in parameter space γ, is a one-to-one mapping. Therefore, hol_{p_o} determines an element of the transformation group of D_{p_o}. This group element is the holonomy of the curve γ.

Often times, the mappings hol_{p_o} only depend on a finite number of parameters. For instance, in the ant example, hol_A was always a linear rotation of vectors. Two-dimensional rotations are, or course, parameterized by a single angle, θ. In these cases, we specify the parameters of the holonomy of a curve rather than the holonomy itself, but make no linguistic distinction between the two.

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