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Weak Turbulence in a Homogeneous Plasma

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MATT-496 November 1966

AEC RESEARCH AND DEVELOPMENT REPORT

This work was supported under Contract AT(30-1)-1238 with the Atomic Energy Commission.

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CHAPTER I

INTRODUCTION

a) Meaning of Turbulence in a Plasma

In recent years considerable effort has been devoted to the study of weakly turbulent plasmas. The term "turbulent" when used to describe the state of a plasma usually refers to situations where a large number of collective modes are excited, for example, through an instability. The label "weak" or "strong" turbulence in general needs clarification in the context of the problem one is considering.

For example, we propose to study the evolution of a spatially homogeneous ensemble of unstable plasmas via the B-B-G-K-Y hierarchy of equations. Three energy densities characterize the problem: the mean particle kinetic energy per unit volume (denote $\mathcal{E}_{\text{kinetic}}$), the energy density of electric field fluctuations associated with the unstable modes (denote $\mathcal{E}_{w}$), and the energy density of the electric field fluctuations that would exist in thermal equilibrium in the absence of instability (denote $\mathcal{E}_{\text{eq. fluctuations}}$). By weak turbulence we mean the case where

$$\mathcal{E}_{\text{eq. fluctuations}} \ll \mathcal{E}_{w} \ll \mathcal{E}_{\text{kinetic}} \quad . \qquad \text{I-1}$$

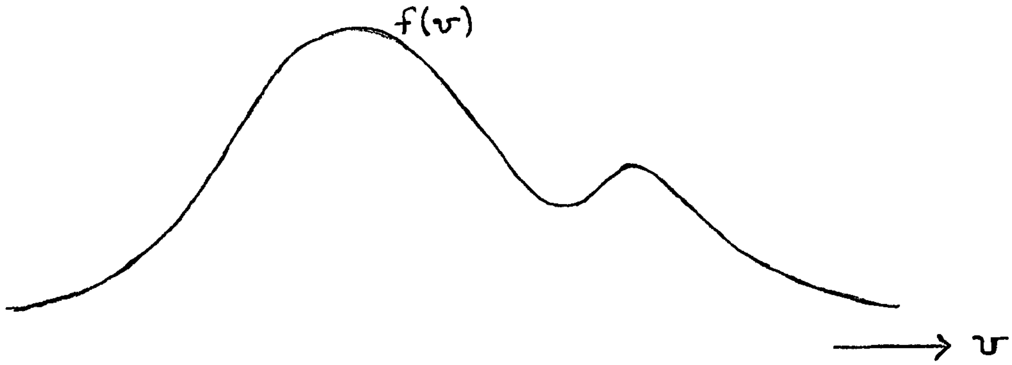
That is, the energy in the field fluctuations is large compared to that which would exist in thermal equilibrium because of the instability (in this sense we have a turbulent situation) but small compared to the mean particle kinetic energy (in this sense the turbulence is "weak"). Strong turbulence would correspond to $\mathcal{E}_{w} \gtrsim \mathcal{E}_{\text{kinetic}}$. The assumption that $\mathcal{E}_{w} \ll \mathcal{E}_{\text{kinetic}}$

essentially limits the growth rate of the instability allowed in the problem. The smallness of the quantity $\mathcal{E}_w/\mathcal{E}_{kinetic}$ makes the problem far more tractable than in the case of strong turbulence since a quasi-linear analysis is possible.

b) Classification of Interactions

We wish at this point to establish some nomenclature. The interactions in a plasma can be more or less classified as wave-particle, particle-particle, and wave-wave. A complete plasma kinetic theory presumably would include all of these effects. However, depending on the particular physical situation, all three types of interactions need not be competing in the sense that they are all dominant effects. For example, one can imagine a spatially homogeneous, stable, plasma (stable in the sense that the zeros of the Landau dielectric function $\epsilon(\underline{k}, s)$, lie in the left-hand s plane) in which the wave-particle and particle-particle interactions play the dominant role. The wave-particle interactions determine the plasma dielectric constant but since the plasma is stable the wave phenomena damp and the system may go to thermal equilibrium through particle-particle interactions before wave-wave interactions are of any significance. On the other hand, if one has an instability present which is driving the electric field fluctuations, as for example in a weakly turbulent plasma, all three processes could be important. However if we visualize the situation where the instability is present but the number of particles in a Debye sphere becomes extremely large (e.g., when the plasma temperature is sufficiently high), then for sufficiently short times the dominant behavior of the plasma may be given by wave-particle and wave-wave interactions since in this case the effective time between collisions (which may be roughly estimated as $\frac{1}{\omega_p}(n\lambda_D^3)$) becomes very long.

We illustrate by a table how the different types of interactions enter the quasi-linear theory of a weakly turbulent plasma with a small electrostatic instability. Keep in mind, we are describing a situation where the Landau dielectric function $\in(k, s)$ allows zeros in the right-half s plane for a band of k, say, as would be the case (in one dimension) for a distribution of the form:



Behavior	Type of Interaction
a) Resonant behavior for $\omega(\underline{k}) - \omega(\underline{\ell}) - \omega(\underline{k} - \underline{\ell}) \sim 0$.	wave-wave
b) Resonant behavior for $\omega(\underline{k}) - \underline{k}\cdot\underline{v} \sim 0$.	wave-particle
c) Resonant behavior for $\omega(\underline{k}) - \omega(\underline{\ell}) - (\underline{k} - \underline{\ell})\cdot\underline{v} \sim 0$.	wave-particle
d) Modification of Balescu-Lenard collision term[1] for a slightly unstable plasma.	particle-particle

Resonant behavior of type (a) will appear in the lowest-order, nonlinear terms in the kinetic equation for the waves. It describes 3-wave processes in which one wave $\omega(\underline{k})$ decouples into two waves $\omega(\underline{\ell})$ and $\omega(\underline{k} - \underline{\ell})$ or the inverse process of 2 waves coupling into one. We term such wave-wave processes as "mode coupling". If the dispersion relation is such that there is no mode coupling, then wave-particle scattering of the form (c) is the dominant nonlinear process.

In reference to (b) one can visualize two physical processes of importance for $\omega(k) - \underline{k} \cdot \underline{v} \sim 0$, namely, resonant diffusion and particle trapping. For simplicity we speak in terms of a one dimensional situation. Existing quasi-linear theories do not include the effect of trapping nor attempt to justify omitting it. Clearly it could be a competing process under certain circumstances. We do not plan to present a quasi-linear theory in which it is an important process but rather attempt to find conditions under which its neglect is justified. At this point we present a simple physical argument to determine such conditions. A more detailed mathematical analysis will be given later in Sec. IV-(g). It is well known[2] that for a charged particle moving approximately at the phase velocity $\omega(k_o)/k_o$, of an electrostatic wave (of amplitude E, wave number k_o), trapping becomes important after a time

$$\tau_{trap} \sim \sqrt{\frac{m}{e E k_o}} . \qquad \text{I-2}$$

The situation we have, however, is that of a packet of waves, say of width Δk, excited in the presence of a weak electrostatic instability. The transit time across the packet of a particle travelling at the phase velocity $v_p \sim \omega(k_o)/k_o$ typical of the waves constituting the packet is given approximately by

$$\tau_{transit} \sim \frac{1}{\Delta k\,(v_p - v_g)} \qquad \text{I-3}$$

where v_g is the group velocity of the packet. If this transit time is short compared to the time τ_{trap} characteristic of the time required by a typical wave of the packet to trap the particle,

$$\text{i.e.,} \quad \tau_{transit} \ll \tau_{trap} \;, \qquad \text{I-4}$$

one would expect trapping to be a relatively unimportant process. The particle diffuses through the packet before it has a chance to be trapped.

The effect of resonant diffusion for $\omega_k - kv \sim 0$ still remains. This can be roughly understood as follows: A particle which travels nearly at the phase velocity of a wave typical of the packet has a long effective interaction period with the wave. This exposure to an almost constant electric field results in an efficient acceleration which carries the particle rapidly away from resonance and results in a large diffusion coefficient in the resonance regime.

Resonance behavior for $\omega_k - \omega_\ell - (k - \ell)v \sim 0$ appears for example, in the nonlinear terms in the kinetic equation for the waves. It is of import to note that the resonant behavior of this form can be associated with nonlinear Landau damping, for although ω_k/k may fall in the region of instability, $\frac{\omega_k - \omega_\ell}{k - \ell}$ may fall in a region of stability.

In reference to (d), Frieman and Rutherford[1] have included, by an appropriate ordering, the effect of particle-particle encounters on the evolution of a weakly turbulent plasma. The usual Lenard-Balescu collision term for the first distribution undergoes some modification through the appearance of certain principle-value integrals. In addition, the kinetic equation for the waves acquires an inhomogeneous source term corresponding to the spontaneous emission of wave energy through particle-particle encounters. However, we reiterate that for sufficiently small plasma parameter, ϵ_p, these effects of single-particle encounters can be pushed to longer and longer times; in this case the weakly turbulent plasma evolves for a considerable length of time through wave-wave and wave-particle interactions of the type (a) - (c).

There is a wide variety of literature[1, 3-11] dealing with the problem of weak turbulence in a fully ionized plasma. Vedenov and Velikhov[3, 4] and Vedenov, Velikhov, and Sagdeev[5] consider only the effects of resonant diffusion of type (b). Drummond and Pines[6] include, in addition to (b), the nonlinear effects of wave-particle scattering on the electric field energy density, thus giving resonant behavior of the form (c). Al'tshul and Karpman,[7] Karpman,[8] Galeev and Karpman[9] and Kadomtsev[10] include in their analyses (b), (c), and the nonlinear effects of wave-wave interactions of type (a), thus giving an additional mechanism for the transfer of energy between modes of different wave number. In all of these treatments,[3-10] the starting point is the magnetic field-free Vlasov equation with self-consistent electric field. In each case the authors solve for the electric field amplitude order by order and then perform an appropriate average over a spatially homogeneous ensemble to obtain a kinetic equation for the electric field energy density.

Iordanskii and Kulikovskii[11] and Frieman and Rutherford,[1] however, have offered an alternate approach. In lieu of working with the Vlasov equation and then performing an average, they operate directly with the B-B-G-K-Y equations for a spatially homogeneous ensemble of slightly unstable plasmas.

Reference 11 includes the effects of wave-particle scattering of the form (b) and (c). Reference 1 deals with (b), (c), and particle-particle effects (d). It is important to note that both in the B-B-G-K-Y hierarchy approach[1, 11] and the Vlasov approach,[3-10] the instability index, $|\gamma_k/\mathrm{Re}\,\omega_k|$, (where $\gamma_k = \mathrm{Im}\,\omega_k$, and $-i\,\omega_k$ is a zero of $\epsilon(\underline{k}, s)$ corresponding to instability) is assumed of the order of some dimensionless small parameter. For large instability indices Iordanskii and Kulikovskii[12] have demonstrated that in the B-B-G-K-Y framework, the large exponential growth makes it necessary to consider the entire chain of equations for the correlation functions, thus rendering the problem mathematically untractable. In fact, the appropriate ordering of the instability index[11, 13] for weak turbulence is $|\gamma_k/\mathrm{Re}\,\omega_k| \sim \epsilon_q \sim \mathcal{E}_w/\mathcal{E}_{kinetic}$.

In this thesis we consider in detail the problem of weak turbulence within the framework of the B-B-G-K-Y hierarchy. We deal with a fully ionized, magnetic field-free, spatially homogeneous ensemble of slightly unstable, multispecies plasmas and derive a coupled pair of equations for the one-particle distribution function f(1), and the spectrumenergy density, $\psi_{\underline{k}}$. The effects of wave-wave interactions (a), wave-particle interactions (b) and (c), are included in the evolution of f(1) and $\psi_{\underline{k}}$. Particle-particle encounters (which, as discussed earlier, can be ordered out of the problem for a considerable length of time if the plasma parameter ϵ_p is sufficiently small) are considered in Chap. IV in relation to the examination of the sign of the spectrum $\psi_{\underline{k}}$.

In Chap. II we consider the hierarchy equations and the weak-turbulence ordering to be employed. With this ordering and utilizing the Bogoliubov and Krylov multiple time scale technique,[14] the equations for the various correlations are established. The solution of these equations resulting in a coupled pair of equations for f(1) and $\psi_{\underline{k}}$ is given in Chap. III. In Chap. IV, various aspects of the final equations of Chap. III are examined. These include associated conservation laws, comparison of the kinetic equation for the spectrum ψ_k with that of Kadomtsev[10] based on a Vlasov model, a consideration of the rapidity of decay of the $e^{-i\underline{k}\cdot\underline{v}t}$ free-streaming terms, comments on the mode coupling (wave-wave) effects, estimating the condition for the neglect of particle trapping effects, and an introduction of the concept of plasmons (quasi-particles).

In Chap. V we consider a model problem exhibiting solely wave-wave interactions.[15, 16] The determining equation is of such a form that lowest-order, linearized version,waves of different wave numbers propagate independently. To next order the nonlinear interactions act as perturbations which slowly transfer energy between modes of different wave number. We use a technique of solution different from references 15 and 16. Whereas they solve the coherent problem order by order and then perform suitable averages over a spatially homogeneous ensemble, we construct at the outset from the original dynamical equation, equations for wave correlations in the ensemble, and obtain from these a kinetic equation describing the evolution of the wave energy density due to wave-wave interactions. This is a very direct way to approach the problem and results in major algebraic simplifications compared to the methods in references 15 and 16, and offers a minimization of conceptual problems.

Using the basic philosophy of Chap. V, in Chap. VI we consider as the dynamical equation, the Vlasov equation with self-consistent electric field. Performing averages over a spatially homogeneous ensemble at the outset (as opposed to references 3-10, where the ensemble averaging is performed after the Vlasov equation is solved) equations for correlations are constructed which are identical to the original B-B-G-K-Y hierarchy equations for g, h,... of Chap. II if the particle-particle terms associated with the discreteness of matter are omitted from the analysis. This result, although not surprising, is significant in at least two aspects. It accounts, for example, for the similarity in the Vlasov (e.g., reference 10) and B-B-G-K-Y hierarchy (Chap. III) weak turbulence results. It also gives strong motivation for employing the B-B-G-K-Y hierarchy in the study of homogeneous turbulence in lieu of the Vlasov approach,[3-10] since, by delaying ensemble averaging, the latter entails much more information (and hence algebra) than necessary.

CHAPTER II

HIERARCHY EQUATIONS AND WEAK TURBULENCE EXPANSION

a) Hierarchy Equations

We consider an ensemble of spatially uniform plasmas described by the B-B-G-K-Y hierarchy of equations.[17] These equations describe the evolution of the s-particle distribution function $f_{a_1 \ldots a_s}(1,2,\ldots,s)$ in terms of the s + 1 particle distribution $f_{a_1 \ldots a_s a_{s+1}}(1,\ldots,s+1)$ and are obtained from the Liouville equation by integrating out the phase-space co-ordinates of the remaining $N-s$ particles. The function $f_{a_1 \ldots a_s}(1,2,\ldots,s)$ describing a multi-component plasma with particles of α kinds, gives the probability density of the joint distribution of s-particles of the types $a_1, a_2, \ldots, a_s$ in the phase space $1,2,\ldots,s$. The notation, $1,2,\ldots,s$ labels the phase space points $(\underline{x}_1, \underline{v}_1)$, $(\underline{x}_2, \underline{v}_2)$, ... We consider these equations in the limit

$$N \to \infty \quad , \quad V/r_0^3 \to \infty \quad ; \quad N r_0^3/V \text{ , finite ,} \tag{II-5}$$

where N is the total number of particles in the system, V the volume of the system, and r_o the typical range of interaction of the particles. This allows us to consider a system of infinite volume and employ a continuous $\underline{k}$ representation.[18] Moreover, we consider in place of the ordinary chain of equations for $f_{a_1 \ldots a_s}(1,2,\ldots,s)$, the chain of equations for the irreducible correlation functions obtained from the distribution function $f_{a_1 \ldots a_s}(1,2,\ldots,s)$ by subtracting from it all possible products of the irreducible correlation function of lower order. We thus have for the second, third, and fourth correlation functions, respectively,

$$g_{a_1 a_2}(1,2) = f_{a_1 a_2}(1,2) - f_{a_1}(1)\, f_{a_2}(2) \tag{II-6}$$

$$h_{a_1a_2a_3} = f_{a_1a_2a_3}(1,2,3) - f_{a_1}(1)\, f_{a_2}(2)\, f_{a_3}(3) - \sum_{\{1,2,3\}} f_{a_1}(1)\, g_{a_2a_3}(2,3) \quad . \qquad \text{II-7}$$

$$k_{a_1a_2a_3a_4}(1,2,3,4) = f_{a_1a_2a_3a_4}(1,2,3,4) - f_{a_1}(1)\, f_{a_2}(2)\, f_{a_3}(3)\, f_{a_4}(4) \qquad \text{II-8}$$
$$- \sum_{\{1,2,3,4\}} f_{a_1}(1) f_{a_2}(2)\, g_{a_3a_4}(3,4) - \sum_{\{1,2,3,4\}} g_{a_1a_2}(1,2)\, g_{a_3a_4}(3,4) - \sum_{\{1,2,3,4\}} f_{a_1}(1)\, h_{a_2a_3a_4}(2,3,4) \; .$$

where $\sum_{\{1,2,\cdots,s\}}$ denotes the sum over cyclic permutations of $\{1,\ldots,s\}$.

The first three members of this hierarchy for a spatially homogeneous ensemble of α species may be written in the absence of a magnetic field

$$\frac{\partial f_{a_1}(1)}{\partial t} = \sum_{a_2=1}^{\alpha} \frac{n_{a_2}}{m_{a_2}} \int \frac{\partial \phi_{a_1a_2}(|\underline{x}_1 - \underline{x}_2|)}{\partial \underline{x}_1} \cdot \frac{\partial}{\partial \underline{v}_1} g_{a_1a_2}(1,2)\, d(2) \qquad \text{II-9}$$

$$\phi_{a_1a_2} = \frac{e_{a_1} e_{a_2}}{|\underline{x}_1 - \underline{x}_2|} \qquad \text{II-10}$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}_2} \right) g_{a_1a_2}(1,2)$$
$$- \frac{1}{m_{a_1}} \frac{\partial f_{a_1}(1)}{\partial \underline{v}_1} \cdot \sum_{a_3=1}^{\alpha} n_{a_3} \int \frac{\partial \phi_{a_1a_3}}{\partial \underline{x}_1} g_{a_2a_3}(2,3)\, d(3)$$
$$- \frac{1}{m_{a_2}} \frac{\partial f_{a_2}(2)}{\partial \underline{v}_2} \cdot \sum_{a_3=1}^{\alpha} n_{a_3} \int \frac{\partial \phi_{a_2a_3}}{\partial \underline{x}_2} g_{a_1a_3}(1,3)\, d(3) \qquad \text{II-11}$$
$$= \frac{\partial \phi_{a_1a_2}}{\partial \underline{x}_1} \cdot \left(\frac{f_{a_2}(2)}{m_{a_1}} \frac{\partial f_{a_1}(1)}{\partial \underline{v}_1} - \frac{f_{a_1}(1)}{m_{a_2}} \frac{\partial f_{a_2}(2)}{\partial \underline{v}_2} \right)$$
$$+ \left\{ \frac{\partial}{\partial \underline{v}_1} \cdot \frac{1}{m_{a_1}} \sum_{a_3=1}^{\alpha} \int \frac{\partial \phi_{a_1a_3}}{\partial \underline{x}_1} h_{a_1a_2a_3}(1,2,3)\, d(3) + (1 \leftrightarrow 2) \right\}$$
$$+ \frac{\partial \phi_{a_1a_2}}{\partial \underline{x}_1} \cdot \left(\frac{1}{m_{a_1}} \frac{\partial}{\partial \underline{v}_1} - \frac{1}{m_{a_2}} \frac{\partial}{\partial \underline{v}_2} \right) g_{a_1a_2}(1,2) \quad .$$

and

$$\left(\frac{\partial}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}_2} + \underline{v}_3 \cdot \frac{\partial}{\partial \underline{x}_3}\right) h_{a_1 a_2 a_3}(1,2,3)$$

$$- \sum_{a_4=1}^{\alpha} \sum_{\{1,2,3\}} \frac{n_{a_4}}{m_{a_1}} \frac{\partial f_{a_1}(1)}{\partial \underline{v}_1} \int \frac{\partial \phi_{a_1 a_4}}{\partial \underline{x}_1} h_{a_2 a_3 a_4}(2,3,4)\, d(4)$$

$$= \sum_{a_4=1}^{\alpha} \sum_{\{1,2,3\}} \left\{ \frac{n_{a_4}}{m_{a_1}} \frac{\partial g_{a_1 a_2}(1,2)}{\partial \underline{v}_1} \cdot \int \frac{\partial \phi_{a_1 a_4}}{\partial \underline{x}_1} g_{a_3 a_4}(3,4)\, d(4) + (2 \leftrightarrow 3) \right\}$$

$$+ \sum_{\{1,2,3\}} \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} \cdot \left(\frac{1}{m_{a_1}} \frac{\partial}{\partial \underline{v}_1} - \frac{1}{m_{a_2}} \frac{\partial}{\partial \underline{v}_2} \right) h_{a_1 a_2 a_3}(1,2,3) \qquad \text{II-12}$$

$$+ \sum_{a_4=1}^{\alpha} \sum_{\{1,2,3\}} \frac{n_{a_4}}{m_{a_1}} \frac{\partial}{\partial \underline{v}_1} \cdot \int \frac{\partial \phi_{a_1 a_4}}{\partial \underline{x}_1} K_{a_1 a_2 a_3 a_4}(1,2,3,4)\, d(4)$$

$$+ \sum_{\{1,2,3\}} \left\{ \frac{\partial \phi_{a_2 a_3}}{\partial \underline{x}_2} \cdot \left(\frac{1}{m_{a_2}} \frac{\partial}{\partial \underline{v}_2} - \frac{1}{m_{a_3}} \frac{\partial}{\partial \underline{v}_3} \right) f_{a_2}(2)\, g_{a_1 a_3}(1,3) + (2 \leftrightarrow 3) \right\}.$$

$$\vdots$$

We do not include any equations for higher correlations since we will achieve closure by ordering out the 4-particle correlation function.

Keep in mind the physical situation we wish to describe, namely, that of a weakly turbulent plasma with

$$\mathcal{E}_{\text{eq. fluctuations}} \ll \mathcal{E}_w \ll \mathcal{E}_{\text{kinetic}}$$

where

$\mathcal{E}_w \sim$ energy density of the fluctuating electric field in the unstable modes

$\mathcal{E}_{\text{eq. fluctuations}} \sim$ energy density of the fluctuating fields in equilibrium

$\mathcal{E}_{\text{kinetic}} \sim$ energy density of kinetic motion of the particles .

Since $\mathcal{E}_{\text{eq. fluctuations}}/\mathcal{E}_{\text{kinetic}} \sim \epsilon_p$ where ϵ_p is the usual plasma parameter

$$\epsilon_p \sim \frac{1}{(n\lambda_D^3)} , \qquad \text{II-13}$$

the above inequality may be written as

$$\epsilon_p \ll \mathcal{E}_w/\mathcal{E}_{\text{kinetic}} \ll 1. \qquad \text{II-14}$$

We introduce the small parameter

$$\epsilon_q \sim \mathcal{E}_w/\mathcal{E}_{\text{KINETIC}} \sim \gamma_{\underline{k}}/\omega_{\underline{k}} , \qquad \text{II-15}$$

where the instability index γ_k/ω_k is typical of the electric fields in the unstable modes.

It is clear that since we wish to describe by the correlation function g the fluctuating fields in the unstable modes with $\mathcal{E}_w \sim \epsilon_q \mathcal{E}_{\text{KINETIC}}$, we have to allow a g which is first order in ϵ_q. This follows from noting (Appendix A) that the energy density of coulomb interaction is a linear, integral functional of g. We are describing a situation in which the correlations are strong compared to the correlations that would exist in equilibrium ($g_{eq} \sim \epsilon_p \ll \epsilon_q$). In order to take full advantage of the smallness of the parameter ϵ_p we estimate the order of magnitude of the terms appearing in Eqs. II-9 - II-12 by introducing

$\langle \phi \rangle \sim$ characteristic strength of the potential,

$r_0 \sim$ effective range of potential,

$v_{av.} \sim$ average particle speed,

$\tau \sim$ typical time scale.

For simplicity we make the estimates for a single species of interacting particles.

The terms in II-9 then stand in the ratio

$$f : \left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right)(n r_o^3)\, g\, v_{av}^3 . \qquad \text{II-16}$$

Similarly II-11 gives

$$\begin{aligned} g &: g\left(\frac{v_{av}\tau}{r_o}\right) : g\left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right)(n r_o^3)\, f\, v_{av}^3 \\ &: \left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right) ff : \left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right)(n r_o^3)\, h\, v_{av}^3 \\ &: \left(\frac{\langle\phi\rangle}{m v_{av}^2}\right)\left(\frac{v_{av}\tau}{r_o}\right) g . \end{aligned} \qquad \text{II-17}$$

A similar analysis of II-12 gives

$$\begin{aligned} h &: h\left(\frac{v_{av}\tau}{r_o}\right) : h\left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right)(n r_o^3)\, f\, v_{av}^3 \\ &: \left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right)(n r_o^3)\, gg\, v_{av}^3 : \left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right) h \\ &: \left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right)(n r_o^3)\, k\, v_{av}^3 : \left(\frac{v_{av}\tau}{r_o}\right)\left(\frac{\langle\phi\rangle}{m v_{av}^2}\right) fg . \end{aligned} \qquad \text{II-18}$$

For our estimate we take r_o , the effective range of the potential, to be of of order the Debye length λ_D and pick as our typical time scale

$$\tau \sim r_o / v_{av} \sim 1/\omega_p , \qquad \text{II-19}$$

then

$$\frac{\langle\phi\rangle}{m v_{av}^2} \sim \frac{1}{n \lambda_D^3} \sim \epsilon_p . \qquad \text{II-20}$$

In addition, we estimate f from the normalization condition $\int f\, d\underline{v} = 1$, and thus obtain $f \sim 1/v_{av}^3$. Estimates II-16, II-17, and II-18 then become

$$f : g\, v_{av}^3 . \qquad \text{II-21}$$

and

$$\begin{aligned} g &: g : g \\ &: \epsilon_p ff : h\, v_{av}^3 \\ &: \epsilon_p g , \end{aligned} \qquad \text{II-22}$$

and also

$$
\begin{aligned}
h &: h : h \\
&: g\, g\, v_{av}^3 : \epsilon_p h \\
&: k\, v_{av}^3 : \epsilon_p f\, q \quad .
\end{aligned}
\qquad \text{II-23}
$$

It is clear we have some liberty in the choice of ϵ_p in relation to ϵ_q (aside from $\epsilon_p << \epsilon_q$). We propose to construct a theory giving a coupled set of equations for $f(1)$ and the electric field fluctuations which involve g to order ϵ_q^2. If we push ϵ_p to smaller and smaller values relative to ϵ_q, for example, by taking the plasma temperature to sufficiently high values we are essentially stretching out the effective collision time between particles $\left(\sim \frac{1}{\epsilon_p \omega_p}\right)$ and would not expect to see the effect of particle-particle interactions in our theory. However, if we take

$$
\epsilon_q^{\ 2} \sim \epsilon_p \qquad \text{II-24}
$$

it is evident from II-22 that in calculating g to order $\epsilon_q^{\ 2}$, the first term in the right of II-11, which gives the usual Lenard-Balescu collision term in the case of a stable plasma, is retained in order $\epsilon_q^{\ 2}$. The obvious advantage of this ordering would be to achieve in order $\epsilon_q^{\ 2}$, competition between wave-particle, particle-particle, and wave-wave interactions. We proceed with the ordering in II-24, keeping in mind that the assumption may be relaxed (so that $\epsilon_p << \epsilon_q^{\ 2}$). As it turns out, the first term on the right-hand side of II-11 will be an exceedingly useful effect to retain when examining the sign of the spectrum of wave energy-density. With $\epsilon_q \sim \epsilon_p^{\ 1/2}$ and the estimates in II-22 and II-23, it is clear that in order to calculate g to order $\epsilon_q^{\ 2}$, we may drop the last term of II-11 and the last three members of II-12.

This follows from examining estimates II-23 and II-24. To lowest order we are assuming $g \sim \epsilon_q$. The gg driving term for h in II-24 thus gives an h of order $\epsilon_q^{\,2}$ in lowest order. The equation for k (not displayed) has gh driving terms, thus giving a k of order $\epsilon_q^{\,3}$. Hence, the last three members of II-23 are of order $\epsilon_q^{\,4}$, $\epsilon_q^{\,3}$, $\epsilon_q^{\,3}$, respectively. However, from II-22, in order to calculate g to order $\epsilon_q^{\,2}$ we only need h to order $\epsilon_q^{\,2}$. We thus omit the last three members of II-12. The last term of II-11 is omitted since it is estimated as order $\epsilon_q^{\,3}$ from II-22. We emphasize that the above estimates have been made by assuming $r_o \sim \lambda_D$. For shorter range collisions terms such as $\frac{\partial \phi}{\partial \underline{x}} \cdot \frac{\partial g}{\partial \underline{v}}$ in II-11 and $\frac{\partial \phi}{\partial \underline{x}} \cdot \frac{\partial h}{\partial \underline{v}}$ in II-12 become important. We do not consider these effects.

With our assumption of spatial homogeneity, we Fourier analyse g(1, 2) in the variables $\underline{x}_1 - \underline{x}_2$ and h(1, 2, 3) in the variables $\underline{x}_1 - \underline{x}_3$ and $\underline{x}_2 - \underline{x}_3$. Equations II-10, II-11, and II-12 become, respectively,

$$\frac{\partial f_{a_1}(1)}{\partial t} = -\sum_{a_2=1}^{\alpha} \frac{4\pi n_{a_2} e_{a_2} e_{a_1}}{m_{a_1}}\, i \int d\underline{k}\, \frac{\underline{k}}{k^2} \cdot \frac{\partial}{\partial \underline{v}_1} \int d\underline{v}_2\, g_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, t) \qquad \text{II-25}$$

$$\left(\frac{\partial}{\partial t} + L_{a_1}(\underline{k}, \underline{v}_1) + L_{a_2}(-\underline{k}, \underline{v}_2) \right) g_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, t)$$

$$= \frac{4\pi e_{a_1} e_{a_2}}{k^2} \left(\frac{f_{a_2}(2)}{m_{a_1}}\, i\underline{k} \cdot \frac{\partial}{\partial \underline{v}_1} f_{a_1}(1) + \begin{pmatrix} 1 \leftrightarrow 2 \\ \underline{k} \to -\underline{k} \end{pmatrix} \right) \qquad \text{II-26}$$

$$+ \left\{ \sum_{a_3=1}^{\alpha} \frac{4\pi n_{a_3} e_{a_3} e_{a_1}}{m_{a_1}} \int \frac{d\underline{l}}{(2\pi)^3} \frac{i(\underline{k} - \underline{l})}{|\underline{k} - \underline{l}|^2} \cdot \frac{\partial}{\partial \underline{v}_1} \int d\underline{v}_3\, h_{a_1 a_2 a_3}(\underline{l}, -\underline{k}, \underline{v}_1, \underline{v}_2, \underline{v}_3, t) \right.$$

$$\left. + \begin{pmatrix} 1 \leftrightarrow 2 \\ \underline{k} \to -\underline{k} \end{pmatrix} \right\}.$$

where

$$L_{a_1}(\underline{k},\underline{v}_1,t) = \iota\underline{k}\cdot\underline{v}_1 - \frac{4\pi e_{a_1}}{m_{a_1}}\frac{\iota\underline{k}}{k^2}\cdot\frac{\partial f_{a_1}}{\partial \underline{v}_1}\sum_{a_1} n_{a_1}e_{a_1}\int d\underline{v}_1 \ , \qquad \text{II-27}$$

Also

$$\left(\frac{\partial}{\partial t} + L_{a_1}(\underline{k}_1,\underline{v}_1,t) + L_{a_2}(\underline{k}_2,\underline{v}_2,t) + L_{a_3}(\underline{k}_3,\underline{v}_3,t)\right) h_{a_1a_2a_3}(\underline{k}_1,\underline{k}_2,\underline{v}_1,\underline{v}_2,\underline{v}_3,t)\,\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3)$$

$$= -\sum_{a_4=1}^{\alpha}\sum_{\{1,2,3\}}\left\{\frac{4\pi n_{a_4}e_{a_4}e_{a_1}}{m_{a_1}}\frac{\iota\underline{k}_3}{k_3^2}\cdot\frac{\partial}{\partial\underline{v}_1} g_{a_2a_1}(\underline{k}_2,\underline{v}_2,\underline{v}_1,t)\int d\underline{v}_4\, g_{a_3a_4}(\underline{k}_3,\underline{v}_3,\underline{v}_4,t) + (2 \leftrightarrow 3)\right\}\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3) \ . \qquad \text{II-28}$$

b) The Expansion

We now expand II-25, II-26, and II-28 in the context of the discussion and estimates presented in the previous section in the small parameter

$$\epsilon_q \sim \epsilon_p^{1/2} \sim \mathcal{E}_w/\mathcal{E}_{\text{KINETIC}} \sim \gamma_k/\omega_k \ ,$$

where γ_k/ω_k is an instability index typical of the fluctuating fields in the unstable modes.

It is clear that at least two time scales, $\tau_o \sim 1/\omega$, where ω is the oscillation frequency of the wave, and $\tau_1 \sim 1/\gamma$ enter the problem. A method for dealing with such asymptotic problems has been given previously by Bogoliubov and Krylov.[14] We expand the distribution and correlation functions in perturbation series

$$f_{a_1} \simeq f_{a_1}^{(o)}(\underline{v}_1, t, \epsilon_q t, \ldots) + \epsilon_q f_{a_1}^{(1)}(\underline{v}_1, t, \epsilon_q t, \ldots) + \ldots \qquad \text{II-29}$$

$$g_{a_1a_2} \simeq \epsilon_q\, g_{a_1a_2}^{(1)}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \epsilon_q t, \ldots) + \epsilon_q^2\, g_{a_1a_2}^{(2)}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \epsilon_q t, \ldots) + \ldots \qquad \text{II-30}$$

$$h_{a_1a_2a_3} \simeq \epsilon_q^2\, h_{a_1a_2a_3}^{(2)}(\underline{k}_1, \underline{k}_2, \underline{v}_1, \underline{v}_2, \underline{v}_3, t, \epsilon_q t, \ldots) + \ldots \qquad \text{II-31}$$

The time variables t, ϵt are treated as independent. The variation of a solution on the ϵt time scale will be determined by removing secular behavior appearing on the t scale, thus rendering the solution uniformly valid. Equations II-25, II-26, and II-28 then become

$$\frac{\partial f_{a_1}^{(0)}}{\partial t} = 0 \qquad \text{II-32}$$

$$\frac{\partial f_{a_1}^{(1)}}{\partial t} + \frac{\partial f_{a_1}^{(0)}}{\partial \epsilon t} = -\sum_{a_2=1}^{\alpha} \frac{4\pi n_{a_2} e_{a_2} e_{a_1}}{m_{a_1}} \int d\underline{k} \frac{i\underline{k}}{k^2} \cdot \frac{\partial}{\partial \underline{v}_1} \int d\underline{v}_2\, g_{a_1 a_2}^{(1)}(\underline{k}, \underline{v}_1, \underline{v}_2, t \cdots) \qquad \text{II-33}$$

$$\frac{\partial f_{a_1}^{(2)}}{\partial t} + \frac{\partial f_{a_1}^{(1)}}{\partial \epsilon t} + \frac{\partial f_{a_1}^{(0)}}{\partial \epsilon^2 t} = -\sum_{a_2=1}^{\alpha} \frac{4\pi n_{a_2} e_{a_2} e_{a_1}}{m_{a_1}} \int d\underline{k} \frac{i\underline{k}}{k^2} \cdot \frac{\partial}{\partial \underline{v}_1} \int d\underline{v}_2\, g_{a_1 a_2}^{(2)}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \epsilon t \cdots) \qquad \text{II-34}$$

$$\vdots$$

$$\left(\frac{\partial}{\partial t} + L_{a_1}^{(0)}(\underline{k}, \underline{v}_1) + L_{a_2}^{(0)}(-\underline{k}, \underline{v}_2) \right) g_{a_1 a_2}^{(1)}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \epsilon t \cdots) = 0 \,, \qquad \text{II-35}$$

$$\left(\frac{\partial}{\partial t} + L_{a_1}^{(0)}(\underline{k}, \underline{v}_1) + L_{a_2}^{(0)}(-\underline{k}, \underline{v}_2) \right) g_{a_1 a_2}^{(2)}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \epsilon t \cdots)$$

$$= -\left(\frac{\partial}{\partial \epsilon t} + L_{a_1}^{(1)}(\underline{k}, \underline{v}_1) + L_{a_2}^{(1)}(-\underline{k}, \underline{v}_2) \right) g_{a_1 a_2}^{(1)} \qquad \text{II-36}$$

$$+ S_{a_1 a_2}^{(2)}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \epsilon t \cdots) + H_{a_1 a_2}^{(2)}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \epsilon t \cdots) \,,$$

$$\vdots$$

$$\delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \left(\frac{\partial}{\partial t} + L_{a_1}^{(0)}(\underline{k}_1, \underline{v}_1) + L_{a_2}^{(0)}(\underline{k}_2, \underline{v}_2) + L_{a_3}^{(0)}(\underline{k}_3, \underline{v}_3) \right) h_{a_1 a_2 a_3}^{(2)}(\underline{k}_1, \underline{k}_2, \underline{v}_1, \underline{v}_2, \underline{v}_3, t \cdots) \qquad \text{II-37}$$

$$= -\delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \sum_{a_4=1}^{\alpha} \sum_{\{1,2,3\}} \left\{ \frac{4\pi n_{a_4} e_{a_4} e_{a_1}}{m_{a_1}} \frac{i\underline{k}_3}{k_3^2} \cdot \frac{\partial}{\partial \underline{v}_1} g_{a_2 a_1}^{(1)}(\underline{k}_2, \underline{v}_2, \underline{v}_1) \int d\underline{v}_4\, g_{a_3 a_4}^{(1)}(\underline{k}_3, \underline{v}_3, \underline{v}_4) \right.$$

$$\left. + (2 \leftrightarrow 3) \right\},$$

$$\vdots$$

where

$$L_{a_1}^{(0)}(\underline{k}, \underline{v}_1) = i\underline{k} \cdot \underline{v}_1 - \frac{4\pi e_{a_1}}{m_{a_1}} \frac{i\underline{k}}{k^2} \cdot \frac{\partial f_{a_1}^{(0)}}{\partial \underline{v}_1} \sum_{a_1} n_{a_1} e_{a_1} \int d\underline{v}_1 \,, \qquad \text{II-38}$$

$$L_{a_1}^{(1)}(\underline{k}, \underline{v}_1) = -\frac{4\pi e_{a_1}}{m_{a_1}} \frac{i\underline{k}}{k^2} \cdot \frac{\partial f_{a_1}^{(1)}}{\partial \underline{v}_1} \sum_{a_1} n_{a_1} e_{a_1} \int d\underline{v}_1 \,, \qquad \text{II-39}$$

and where

$$S^{(2)}_{a_1 a_2} = \frac{4\pi e_{a_1} e_{a_2}}{k^2} i\underline{k} \cdot \left(\frac{f^{(0)}_{a_2}(2)}{m_{a_1}} \frac{\partial}{\partial \underline{v}_1} f^{(0)}_{a_1}(1) - (1 \leftrightarrow 2) \right), \qquad \text{II-40}$$

and

$$H^{(2)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \cdots)$$
$$= \left\{ \sum_{a_3=1}^{\alpha} \frac{4\pi n_{a_3} e_{a_3} e_{a_1}}{m_{a_1}} \int \frac{d\underline{\ell}}{(2\pi)^3} \frac{i(\underline{k}-\underline{\ell})}{|\underline{k}-\underline{\ell}|^2} \cdot \frac{\partial}{\partial \underline{v}_1} \int d\underline{v}_3 \, h^{(2)}_{a_1 a_2 a_3}(\underline{\ell}, -\underline{k}, \underline{v}_1, \underline{v}_2, \underline{v}_3 \right.$$
$$\left. + \begin{pmatrix} 1 \leftrightarrow 2 \\ \underline{k} \rightarrow -\underline{k} \end{pmatrix} \right\}.$$

We reiterate that our basic aim in the analysis of these equations is to obtain a coupled pair of equations for the first distribution f(1) and the energy density of the fluctuating fields. We proceed under the assumption that the time scales relevant to the problem are $\tau_0 = t_1, \tau_1 = \epsilon_q t, \ldots$ In the analysis which follows, poles associated with free-streaming behavior of the form $e^{-i\underline{k}\cdot\underline{v}t}$ are consistently omitted under the assumption that these effects phase-mix out as far as velocity integrals are concerned. The rapidity of decay of typical free streaming effects is briefly considered in Chap. IV and also the condition I-4 for neglecting the effects of particle trapping is obtained.

CHAPTER III

SOLUTION

We now turn to solving Eqs. II-32 - II-37 starting with the lowest-order equations. Since $f^{(0)}_{a_1}(1)$ does not vary on the t scale by virtue of Eq. II-32, the linear operators $L^{(0)}_{a}(\underline{k},\underline{v})$ are independent of t and Eq. II-35 may be most easily solved by means of Laplace transforming in t to give (see Appendix B)

$$g^{(1)}_{a_1a_2}(\underline{k},\underline{v}_1,\underline{v}_2,t,\epsilon t,\cdots) = \frac{1}{(2\pi i)^2}\int_{C_1}\int_{C_2}\frac{ds_1\,ds_2\, e^{s_1 t}\, e^{s_2 t}}{(s_1 + L^{(0)}_{a_1}(\underline{k},\underline{v}_1))(s_2 + L^{(0)}_{a_2}(-\underline{k},\underline{v}_2))}\, g_{a_1a_2}(\underline{k},\underline{v}_1,\underline{v}_2,0,\epsilon t), \qquad \text{III-42}$$

where

$$\frac{1}{s_1 + L^{(0)}_{a_1}(\underline{k},\underline{v}_1)} \equiv \frac{1}{s_1 + i\underline{k}\cdot\underline{v}_1}\left(1 + \frac{4\pi e_{a_1}}{m_{a_1}\epsilon(\underline{k},s_1)}\frac{i\underline{k}}{k^2}\cdot\frac{\partial f^{(0)}_{a_1}}{\partial\underline{v}_1}\sum_{a_1=1}^{\alpha} n_{a_1}e_{a_1}\int\frac{d\underline{v}_1}{s_1 + i\underline{k}\cdot\underline{v}_1}\right) \qquad \text{III-43}$$

and

$$\epsilon(\underline{k},s) = 1 - \sum_{a_1=1}^{\alpha}\frac{4\pi n_{a_1}e_{a_1}^2\, i}{m_{a_1}}\int\frac{\underline{k}\cdot\partial f^{(0)}_{a_1}/\partial\underline{v}_1\, d\underline{v}_1}{s + i\underline{k}\cdot\underline{v}_1}.$$

In reference to III-42, C_1 and C_2 are the usual Laplace contours, parallel to the Im s_1 (Im s_2) axis and to the right of singularities of the integrand. We assume as in Ref. 1 that $\epsilon(\underline{k}, s) = 0$ admits marginally stable solutions for a band of $\underline{k}$. In the interesting case of an electron gas in a smeared-out, neutralizing background of fixed ions, $f^{(0)}_{e}$ is represented schematically in Diag. 1. The function $f^{(1)}_{e}$ represents the small bump in the distribution which yields the instability of order ϵ. The situation in a multicomponent plasma is clearly more complex; however, for completeness we allow for such a possibility. We assume that for a given $\underline{k}$ (in the region of $\underline{k}$-space corresponding to marginally stable solutions of $\epsilon(\underline{k}, s) = 0$) there is a single marginally stable mode $s_{\underline{k}} = -i\omega_{\underline{k}}$. For $\underline{k}$ outside of this region as well as inside there will be solutions of $\epsilon(\underline{k}, s) = 0$ corresponding to Landau damping. As phenomena associated with these solutions damp, we do not include them in the present analysis. We assume their contribution is negligible in times of order $1/\epsilon_q \omega_k$. Relation III-42 is a purely formal solution to the problem. However, by virtue of our knowledge of the solution to the Landau problem we know the form of the operator $\frac{1}{s + L_{a_i}}$, namely, that given in expression III-43. When operator III-43 acts on some initial value $f_{\underline{k}}(\underline{v}_1, 0)$ which is analytic in a sufficiently broad strip containing $\underline{v}_1$ real, and its Laplace inversion taken, it gives rise to $e^{-i\underline{k}\cdot\underline{v}_1 t}$, and $e^{-i\omega_k t}$ terms as $t \to \infty$. Similarly for $g^{(1)}_{a_1 a_2}(\underline{v}_1\, \underline{v}_2, 0, \epsilon t)$ analytic in sufficiently broad strips containing $\underline{v}_1$, $\underline{v}_2$ real, the solution III-42 will involve products of terms involving $e^{-i\underline{k}\cdot\underline{v}_1 t}$, $e^{-i\omega_k t}$ and $e^{-i\underline{k}\cdot\underline{v}_2 t}$, $e^{i\omega_k t}$.

The philosophy we adopt is that of treating g as a distribution in velocities $\underline{v}_1$ and $\underline{v}_2$ in the sense of Schwartz, i. e., treat it as a distribution for calculating moments with respect to $\underline{v}_1$ and $\underline{v}_2$. For instance, as discussed in Appendix A, the quantity

$$\frac{1}{8\pi k^2} \sum_{a_1, a_2} (4\pi n_{a_1} e_{a_1})(4\pi n_{a_2} e_{a_2}) \iint d\underline{v}_1\, d\underline{v}_2\, g_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, t), \qquad \text{III-44}$$

which involves the simple moment $\iint d\underline{v}_1\, d\underline{v}_2\, g_{a_1 a_2}$, is a meaningful physical entity to consider since it represents the energy density of coulomb interaction. Similarly Eq. II-25 which advances $f_{a_1}(\underline{v}_1, t)$ involves the integral $\int d\underline{v}_2\, g_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, t, \ldots)$. If we use this equation for determining the time evolution of the moment of a function $\oplus(\underline{v}_1)$ over f, then integrating by parts and assuming that $\oplus(\underline{v}_1)\, g_{a_1 a_2}(v_1, v_2, t)$ vanishes sufficiently rapidly for large $|\underline{v}_{i}|$,

$$\frac{\partial}{\partial t}\int \oplus(\underline{v}_1) f_{a_1}(\underline{v}_1,t)\,d\underline{v}_1 = \sum_{a_2=1}^{\alpha} \frac{4\pi n_{a_2} e_{a_2} e_{a_1} i}{m_{a_1}} \int d\underline{k}\, \frac{\underline{k}}{k^2}\cdot \int d\underline{v}_1 d\underline{v}_2 \frac{\partial \oplus}{\partial \underline{v}_1}\, g_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2), \qquad \text{III-45}$$

where we have interchanged the order of the $\underline{k}$ and $\underline{v}_1$ integrations. We see that Eq. III-45 involves the moment

$$\iint d\underline{v}_1\, d\underline{v}_2 \frac{\partial \oplus(\underline{v}_1)}{\partial \underline{v}_1}\, g_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2,t)\,.$$

With the assumption that $\oplus(\underline{v}_1)$ and $g_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2,t)$ are analytic in sufficiently broad strips containing $\underline{v}_1$ and $\underline{v}_2$ real, the terms involving $e^{-i\underline{k}\cdot\underline{v}_1 t}$ and $e^{i\underline{k}\cdot\underline{v}_2, t}$ in $g^{(1)}_{a_1 a_2}$ decay as $t \rightarrow \infty$ by a phase-mixing process. The lowest-order correlation function $g^{(1)}_{a_1 a_2}$ is then effectively given by the product involving $e^{-i\omega_k t}$ and $e^{i\omega_k t}$. For large t, $g^{(1)}_{a_1 a_2}$ as a distribution in $\underline{v}_1$ and $\underline{v}_2$ can be written as (see Appendix C_1)

$$g^{(1)}_{a_1 a_2} = \frac{e_{a_1}}{m_{a_1}} \frac{i\underline{k}\cdot \partial f^{(0)}_{a_1}/\partial \underline{v}_1}{-i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}_1 + \Delta} \frac{e_{a_2}}{m_{a_2}} \frac{-i\underline{k}\cdot\partial f^{(0)}_{a_2}/\partial \underline{v}_2}{i\omega_{\underline{k}} - i\underline{k}\cdot\underline{v}_2 + \Delta}\, \psi_{\underline{k}} \qquad \text{III-46}$$

where ψ_k is $\frac{8\pi}{k^2}$ times the lowest order energy density of coulomb interactions associated with $g^{(1)}_{a_1 a_2}$ (see Eq. III-44) and is given by

$$\psi_{\underline{k}} = \frac{1}{k^4} \sum_{a_1,a_2} \frac{(4\pi n_{a_1} e_{a_1})(4\pi n_{a_2} e_{a_2})}{\left|\frac{\partial \varepsilon_k}{\partial s}\right|^2_{s=-i\omega_k}} \iint \frac{d\underline{v}_1\, d\underline{v}_2\, g^{(1)}_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2,0,\text{etc.})}{(-i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}_1+\Delta)(i\omega_{\underline{k}} - i\underline{k}\cdot\underline{v}_2+\Delta)} \cdot \qquad \text{III-47}$$

The rapidity of decay of the free-streaming terms, the question as to whether the $\underline{k}$ integrals (as in Eq. III-45)) over the decaying terms associated with the velocity integral of the streaming terms $e^{-i\underline{k}\cdot\underline{v}t}$ themselves decay rapidly enough, the neglect of particle trapping, are questions important to the validity of the theory. We consider these issues in some detail in the next chapter. There, conditions on the width of the $\underline{k}$ spectrum excited, group velocity restrictions in relation to the wave-wave terms, etc., are obtained.

We do not pursue them further at this point.

When expression III-46 is substituted into Eq. II-33, one readily finds for large t

$$\frac{\partial f_{a_1}^{(1)}}{\partial t} + \frac{\partial f_{a_1}^{(0)}}{\partial \epsilon t} = -i\left(\frac{e_{a_1}}{m_{a_1}}\right)^2 \int d\underline{k}\; \underline{k}\cdot\frac{\partial}{\partial \underline{v}_1}\; \frac{-\underline{k}\cdot\partial f_{a_1}^{(0)}/\partial \underline{v}_1}{\omega_{\underline{k}} - \underline{k}\cdot\underline{v}_1 + i\Delta}$$

$$\times\left(1 - \epsilon(-\underline{k}, i\omega_{\underline{k}} + \Delta)\right)\psi_{\underline{k}} \quad . \qquad \text{III-48}$$

Since $\psi_{-\underline{k}} = \psi_{\underline{k}}$, this simply reduces to

$$\frac{\partial f_{a_1}^{(1)}}{\partial t} + \frac{\partial f_{a_1}^{(0)}}{\partial \epsilon t} = \left(\frac{e_{a_1}}{m_{a_1}}\right)^2 \int d\underline{k}\; \underline{k}\cdot\frac{\partial}{\partial \underline{v}_1}\left\{\pi\,\delta(\omega_{\underline{k}} - \underline{k}\cdot\underline{v}_1)\,\underline{k}\cdot\frac{\partial f_{a_1}^{(0)}}{\partial \underline{v}_1}\right\}\psi_{\underline{k}} \; .$$

As $f_{a_1}^{(0)}$ and $\psi_{\underline{k}}$ are independent of t, in order that $f_{a_1}^{(1)}$ need not be secular, we ask that

$$\frac{\partial f_{a_1}^{(1)}}{\partial t} = 0 \quad , \qquad \text{III-49}$$

and

$$\frac{\partial f_{a_1}^{(0)}}{\partial \epsilon t} = \left(\frac{e_{a_1}}{m_{a_1}}\right)^2 \int d\underline{k}\; \underline{k}\cdot\frac{\partial}{\partial \underline{v}_1}\left\{\pi\,\delta(\omega_{\underline{k}} - \underline{k}\cdot\underline{v}_1)\,\underline{k}\cdot\frac{\partial f_{a_1}^{(0)}}{\partial \underline{v}_1}\right\}\psi_{\underline{k}} \; . \qquad \text{III-50}$$

In reference to Eq. III-50, we assume that the dependence of $\omega_{\underline{k}}$ on ϵt is given by $\epsilon(\underline{k}, -i\omega_{\underline{k}} + \Delta) = 0$, an integral equation relating $\omega_{\underline{k}}$ and $f_a^{(0)}$. The time evolution of $f_a^{(0)}$ on the ϵt scale given by Eq. III-50 is in no sense trivial in the general three-dimensional problem. However, in the case of a one-dimensional gas of electrons embedded in a fixed neutralizing background of positive ions, then III-50, in conjunction with $\epsilon(k, -i\omega_k + \Delta) = 0$, tells us that $\frac{\partial f_{a_1}^{(0)}}{\partial \epsilon t} = 0$. As discussed in Chap. IV, III-50 does have the desirable property that it conserves probability ($\int f_a^{(0)}\, d\underline{v}$), total particle momentum

$\sum_a n_a \int f_a^{(0)} m_a \underline{v}\, d\underline{v}$, and total particle kinetic energy $\sum_a n_a \int f_a^{(0)} \frac{m_a v^2}{2} d\underline{v}$, on the ϵt scale. In order to solve Eq. II-36 for $g^{(2)}_{a_1 a_2}$, an expression for $H^{(2)}_{a_1 a_2}$ and hence $h^{(2)}_{a_1 a_2 a_3}$ is needed. In reference to the solution of Eq. II-37, we introduce the notation (see expression III-46)

$$g^{(1)}_{a_1 a_2} = g_{a_1}(\underline{k}, \underline{v}_1)\; g_{a_2}(-\underline{k}, \underline{v}_2)\; \psi_{\underline{k}} , \qquad \text{III-51}$$

where

$$g_{a_1}(\underline{k}, \underline{v}_1) = \frac{e_{a_1}}{m_{a_1}} \; \frac{i\underline{k} \cdot \partial f^{(0)}_{a_1} / \partial \underline{v}_1}{-i\omega_{\underline{k}} + i\underline{k} \cdot \underline{v}_1 + \Delta} .$$

Substituting the $t \to \infty$ expression for $g^{(1)}_{a_1 a_2}$ given by III-51 into the right-hand side of Eq. II-37 readily gives

$$\begin{aligned} &\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3)\left(\frac{\partial}{\partial t} + L^{(0)}_{a_1}(\underline{k}_1,\underline{v}_1) + L^{(0)}_{a_2}(\underline{k}_2,\underline{v}_2) + L^{(0)}_{a_3}(\underline{k}_3,\underline{v}_3)\right) h^{(2)}_{a_1 a_2 a_3}(\underline{k}_1,\underline{k}_2,\underline{v}_1,\underline{v}_2,\underline{v}_3,t,\epsilon t \cdots) \\ &= -\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3) \sum_{\{1,2,3\}} \Big\{ \frac{e_{a_1}}{m_{a_1}} i\underline{k}_3 \cdot \frac{\partial}{\partial \underline{v}_1} g_{a_2}(\underline{k}_2,\underline{v}_2)\, g_{a_1}(-\underline{k}_2,\underline{v}_1)\, g_{a_3}(\underline{k}_3,\underline{v}_3)\, \psi_{\underline{k}_2} \psi_{\underline{k}_3} \qquad \text{III-52} \\ &\qquad + (2 \leftrightarrow 3) \Big\} \end{aligned}$$

where we have used the fact that

$$\sum_{a_4} \frac{4\pi n_{a_4} e_{a_4}^2}{m_{a_4} k_3^2} \int - \frac{i\underline{k}_3 \cdot \partial f^{(0)}_{a_4} / \partial \underline{v}_4 \, d\underline{v}_4}{i\omega_{\underline{k}_3} - i\underline{k}_3 \cdot \underline{v}_4 + \Delta} = 1 , \qquad \text{III-53}$$

i. e.,

$$\begin{aligned} &\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3)\left(\frac{\partial}{\partial t} + L^{(0)}_{a_1}(\underline{k}_1,\underline{v}_1) + L^{(0)}_{a_2}(\underline{k}_2,\underline{v}_2) + L^{(0)}_{a_3}(\underline{k}_3,\underline{v}_3)\right) h^{(2)}_{a_1 a_2 a_3} \qquad \text{III-54} \\ &= -\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3) \sum_{\{1,2,3\}} g_{a_2}(\underline{k}_2,\underline{v}_2)\, g_{a_3}(\underline{k}_3,\underline{v}_3)\, \psi_{\underline{k}_2} \psi_{\underline{k}_3} \times \\ &\qquad \Big\{ \frac{e_{a_1}}{m_{a_1}} i\underline{k}_3 \cdot \frac{\partial}{\partial \underline{v}_1} g_{a_1}(-\underline{k}_2,\underline{v}_1) + \frac{e_{a_1}}{m_{a_1}} i\underline{k}_2 \cdot \frac{\partial}{\partial \underline{v}_1} g_{a_1}(-\underline{k}_3,\underline{v}_1) \Big\} . \end{aligned}$$

The formal solution to this may be written as

$$\begin{aligned} &h^{(2)}_{a_1 a_2 a_3}(\underline{k}_1,\underline{k}_2,\underline{v}_1,\underline{v}_2,\underline{v}_3,t,\epsilon t,\cdots)\, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3) \\ &= \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3)\, e^{-(L_{a_1}+L_{a_2}+L_{a_3})t}\, h^{(2)}_{a_1 a_2 a_3}(\underline{k}_1,\underline{k}_2,\underline{v}_1,\underline{v}_2,\underline{v}_3,0,\epsilon t, \cdots) \\ &\quad - \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3) \sum_{\{1,2,3\}} e^{-(L_{a_1}+L_{a_2}+L_{a_3})t} \int_0^t dt'\, e^{(L_{a_1}+L_{a_2}+L_{a_3})t'} \times \qquad \text{III-55} \\ &\quad \times g_{a_2}(\underline{k}_2,\underline{v}_2)\, g_{a_3}(\underline{k}_3,\underline{v}_3)\, \psi_{\underline{k}_2} \psi_{\underline{k}_3} \Big\{ \frac{e_{a_1}}{m_{a_1}} i\underline{k}_3 \cdot \frac{\partial}{\partial \underline{v}_1} g_{a_1}(-\underline{k}_2,\underline{v}_1) + (2 \leftrightarrow 3) \Big\} , \end{aligned}$$

where L_{a_i} is short form for $L^{(o)}_{a_i}(\underline{k}_i, \underline{v}_i)$. In Laplace variables (see Appendix C_2) apart from the initial value term, the right-hand side of Eq. II-41 is simply

$$-\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3)\sum_{\{1,2,3\}}\frac{g_{a_2}(\underline{k}_2,\underline{v}_2)\,g_{a_3}(\underline{k}_3,\underline{v}_3)\,\Psi_{\underline{k}_2}\Psi_{\underline{k}_3}}{s(s+i\omega_{\underline{k}_2}+i\omega_{\underline{k}_3}+L^{(o)}_{a_1}(\underline{k}_1,\underline{v}_1))}\times$$
$$\left\{\frac{e_{a_1}}{m_{a_1}}i\underline{k}_3\cdot\frac{\partial}{\partial\underline{v}_1}g_{a_1}(-\underline{k}_2,\underline{v}_1)+(2\leftrightarrow 3)\right\}. \qquad \text{III-56}$$

The only effect of the initial value $h^{(2)}_{a_1a_2a_3}(o, \epsilon t, \ldots)$ is to give a contribution to $g^{(2)}_{a_1a_2}$ (through the evaluation of $H^{(2)}_{a_1a_2}$ in Eq. II-36) of the form

$$g^{(2)}_{a_1a_2} = g_{a_1}(\underline{k},\underline{v}_1)\; g_{a_2}(-\underline{k},\underline{v}_2)\;\tilde{\Psi}_{\underline{k}} \quad , \qquad \text{III-57}$$

which is identical in structure to $g^{(1)}_{a_1a_2}$ given in Eq. III-46 and can be trivially absorbed into the result. This is briefly discussed in Appendix C_3 .

In reference to III-56 we introduce the notation

$$G_{a_1}(-\underline{k}_2-\underline{k}_3,\underline{v}_1,s) = \frac{-1}{(s+i\omega_{\underline{k}_2}+i\omega_{\underline{k}_3}+L^{(o)}_{a_1}(\underline{k}_1,\underline{v}_1))}\left\{\frac{e_{a_1}}{m_{a_1}}i\underline{k}_3\cdot\frac{\partial}{\partial\underline{v}_1}g_{a_1}(-\underline{k}_2,\underline{v}_1)\right. \qquad \text{III-58}$$
$$\left.+(2\leftrightarrow 3)\right\}.$$

In order to calculate $g^{(2)}_{a_1a_2}$ for large t we need to evaluate the driving term $H^{(2)}_{a_1a_2}$ where

$$H^{(2)}_{a_1a_2}(\underline{k},\underline{v}_1,\underline{v}_2,s)$$
$$=\sum_{a_3=1}^{\alpha}\frac{e_{a_1}}{m_{a_1}}4\pi n_{a_3}e_{a_3}\int\frac{d\underline{l}}{(2\pi)^3}\frac{i(\underline{k}-\underline{l})}{|\underline{k}-\underline{l}|^2}\cdot\frac{\partial}{\partial\underline{v}_1}\int d\underline{v}_3\, h^{(2)}_{a_1a_2a_3}(\underline{l},-\underline{k},\underline{v}_1,\underline{v}_2,\underline{v}_3,s,\epsilon t\cdots)$$
$$+\begin{pmatrix}\underline{k}\to-\underline{k}\\ 1\leftrightarrow 2\end{pmatrix}. \qquad \text{III-59}$$

Utilizing the notation in III-58, and that

$$\sum_{a_3}\frac{4\pi n_{a_3}e_{a_3}}{|\underline{k}-\underline{l}|^2}\int d\underline{v}_3\; g_{a_3}(\underline{k}-\underline{l},\underline{v}_3) = 1 \quad ,$$

III-59 readily becomes

$$H^{(2)}_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2,s)$$

$$= \frac{1}{s}\frac{e_{a_1}}{m_{a_1}}\Big\{ g_{a_2}(-\underline{k},\underline{v}_2)\int \frac{d\underline{\ell}}{(2\pi)^3} L(\underline{k}-\underline{\ell})\cdot\frac{\partial}{\partial \underline{v}_1}\Big[\psi_{\underline{k}}\psi_{\underline{k}-\underline{\ell}}\, G_{a_1}(\underline{k},\underline{\ell}-\underline{k},\underline{v}_1,s)$$

$$+\psi_{\underline{\ell}}\psi_{\underline{k}}\, g_{a_1}(\underline{\ell},\underline{v}_1)\sum_{a_3}\frac{4\pi n_{a_3}e_{a_3}}{|\underline{k}-\underline{\ell}|^2}\int d\underline{v}_3\, G_{a_3}(-\underline{\ell},\underline{k},\underline{v}_3,s)\Big]$$

$$+\int\frac{d\underline{\ell}}{(2\pi)^3}\psi_{\underline{\ell}}\psi_{\underline{k}-\underline{\ell}}\, L(\underline{k}-\underline{\ell})\cdot\frac{\partial}{\partial\underline{v}_1} G_{a_2}(\underline{\ell}-\underline{k},\underline{\ell},\underline{v}_2,s)\, g_{a_1}(\underline{\ell},\underline{v}_1)\Big\}$$

$$+\begin{pmatrix}\underline{k}\to-\underline{k}\\ 1\leftrightarrow 2\end{pmatrix}. \qquad \text{III-60}$$

The last term in Eq. III-60 involving $\psi_{\underline{\ell}}\ \psi_{\underline{k}-\underline{\ell}}$ bears special significance in that it will lead to mode coupling terms (involving resonant behavior for $\omega_{\underline{k}} - \omega_{\underline{\ell}} - \omega_{\underline{k}-\underline{\ell}} \backsim 0$) in the kinetic equation for the waves. The first two terms lead to particle-wave resonant behavior (for $\omega_{\underline{k}} - \omega_{\underline{\ell}} - (\underline{k}-\underline{\ell})\cdot\underline{v} \backsim 0$) as well as mode coupling phenomena in the kinetic equation for the waves. The contribution of $H^{(2)}_{a_1a_2}$ to $g^{(2)}_{a_1a_2}$ (denote $g^{(2)}_{a_1a_2}(H)$) can be written formally from Eq. II-36 as

$$g^{(2)}_{a_1a_2}(H)$$

$$= \frac{1}{s + L^{(0)}_{a_1}(\underline{k},\underline{v}_1) + L^{(0)}_{a_2}(-\underline{k},\underline{v}_2)}\ H^{(2)}_{a_1a_2}(\underline{k},\underline{v}_1,\underline{v}_2,s)$$

$$= \int_{C_1}\int_{C_2}\frac{ds_1\,ds_2/(2\pi i)^2}{s-s_1-s_2}\ \frac{1}{s_1+L^{(0)}_{a_1}(\underline{k},\underline{v}_1)}\ \frac{1}{s_2+L^{(0)}_{a_2}(-\underline{k},\underline{v}_2)}\ H^{(2)}_{a_1a_2}\,. \qquad \text{III-61}$$

Since (Appendix C_2)

$$\frac{1}{s_2+L^{(0)}_{a_2}(-\underline{k},\underline{v}_2)}\ g_{a_2}(-\underline{k},\underline{v}_2) = \frac{g_{a_2}(-\underline{k},\underline{v}_2)}{s_2 - i\omega_{\underline{k}}},$$

the integration in the first two terms of III-60 may be done trivially first by closing C_2 to the left and then by closing C_1 to the right; the result is readily

$$g^{(2)}_{a_1 a_2}(H)$$

$$= \frac{g_{a_2}(-\underline{k},\underline{v}_2)}{s\,(s - i\omega_{\underline{k}} + L^{(0)}_{a_1}(\underline{k},\underline{v}_1))} \frac{e_{a_1}}{m_{a_1}} \Big\{ \int \frac{d\underline{\ell}}{(2\pi)^3}\, i(\underline{k}-\underline{\ell})\cdot\frac{\partial}{\partial \underline{v}_1} \Big[\psi_{\underline{k}}\, \psi_{\underline{k}-\underline{\ell}}\, G_{a_1}(\underline{k},\underline{\ell}-\underline{k},\underline{v}_1,s)$$

$$+ \psi_{\underline{\ell}}\, \psi_{\underline{k}}\, g_{a_1}(\underline{\ell},\underline{v}_1) \sum_{a_3} \frac{4\pi n_{a_3} e_{a_3}}{|\underline{k}-\underline{\ell}|^2} \int d\underline{v}_3\, G_{a_3}(-\underline{\ell},\underline{k},\underline{v}_3,s) \Big] \Big\}$$

$$+ \int_{C_1}\int_{C_2} \frac{ds_1\, ds_2\, /(2\pi i)^2}{s\,(s-s_1-s_2)} \frac{1}{s_1 + L^{(0)}_{a_1}(\underline{k},\underline{v}_1)} \frac{1}{s_2 + L^{(0)}_{a_2}(-\underline{k},\underline{v}_2)} \times$$

$$\times \frac{e_{a_1}}{m_{a_1}} \int \frac{d\underline{\ell}}{(2\pi)^3} \psi_{\underline{\ell}}\, \psi_{\underline{k}-\underline{\ell}}\, i(\underline{k}-\underline{\ell})\cdot\frac{\partial}{\partial \underline{v}_1} G_{a_2}(\underline{\ell}-\underline{k},\underline{\ell},\underline{v}_2,s)\, g_{a_1}(\underline{\ell},\underline{v}_1) \quad + \begin{pmatrix} \underline{k}\to-\underline{k} \\ 1\leftrightarrow 2 \end{pmatrix} \qquad \text{III-62}$$

Similarly the contribution to $g^{(2)}_{a_1 a_2}$ from the first three driving terms on the right-hand side of Eq. II-36 can be written, upon inserting expression III-51 for $g^{(1)}_{a_1 a_2}$ and using Eq. III-44, as

$$g^{(2)}_{a_1 a_2} = \frac{-\,g_{a_2}(-\underline{k},\underline{v}_2)}{s\,(s - i\omega_{\underline{k}} + L^{(0)}_{a_1}(\underline{k},\underline{v}_1))} \Big[\psi_{\underline{k}} \frac{\partial}{\partial \epsilon t} g_{a_1}(\underline{k},\underline{v}_1) + \frac{1}{2} g_{a_1}(\underline{k},\underline{v}_1) \frac{\partial}{\partial \epsilon t} \psi_{\underline{k}} - \frac{e_{a_1}}{m_{a_1}}\, i\underline{k}\cdot\frac{\partial}{\partial \underline{v}_1} f^{(1)}_{a_1}\, \psi_{\underline{k}} \Big]$$

$$+ \begin{pmatrix} \underline{k} \to -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix}, \qquad \text{III-63}$$

where we have expressed the result in the form $(\underline{k}, 1, 2)$ plus the $\begin{pmatrix} \underline{k} \to -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix}$ version of the same to concur with the symmetry in that portion of $g^{(2)}_{a_1 a_2}$ driven by $H^{(2)}_{a_1 a_2}$ given in Eq. III-62.

We do not include the effects of the source term $S^{(2)}_{a_1 a_2}$ at this point. As mentioned earlier this can be ordered out of the problem for sufficiently small ϵ_p, i.e., for sufficiently long effective time between collisions. The phenomena we examine then effectively evolve in times shorter than the collision time. The source term will prove a useful effect to include when examining the sign of the spectrum $\psi_{\underline{k}}$. We postpone its discussion until a later time.

It is clear that the initial value of $g^{(2)}_{a_1a_2}$ on the t scale, namely, $g^{(2)}_{a_1a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, 0, \epsilon t)$, will lead to a solution identical in form to that of $g^{(1)}_{a_1a_2}$ given in Eq. III-51 and can be trivially absorbed into the same. As it leads to no new information we omit it from the present calculation. In order to determine the time evolution of the one-particle distribution from Eq. II-34 we need the inversion of the net expression for $g^{(2)}_{a_1a_2}(s)$ given by III-62 and III-63 for large t. There will be a secular portion varying as t as well as a steady asymptotic part. The secularity arises from the double pole at $s = 0$ in the factor $\frac{1}{s} \frac{1}{\epsilon(\underline{k}, s - i\omega_k)}$ occurring in the operator $\frac{1}{s(s - i\omega_k + L_{a_1})}$ (as well as its $\begin{pmatrix} \underline{k} \rightarrow -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix}$ version). The removal of the secular behavior will then yield an equation giving the time evolution of the spectrum ψ_k on the ϵt scale. The nonsecular part of $g^{(2)}_{a_1a_2}$ which remains is to be used in Eq. II-34 to determine the evolution of the first distribution function. The analysis of the third and sixth terms in III-62 will demand special attention. In fact in situations where

$$\omega(\underline{k}) - \omega(\underline{\ell}) = \omega(\underline{k} - \underline{\ell}) \qquad \text{III-64}$$

can occur it yields a secular contribution to $g^{(2)}_{a_1a_2}$ and hence a driving term in the kinetic equation for $\psi_{\underline{k}}$.

The evaluation of $g^{(2)}_{a_1a_2}$ is tedious but straightforward and is given in Appendix D and E. In Appendix D we consider the contribution to $g^{(2)}_{a_1a_2}$ by the third and sixth terms of Eq. III-62, whereas in Appendix E the remaining terms in III-62 and III-63 are examined. It is of considerable import to note that there is a direct way of obtaining the kinetic equation for the spectrum $\psi_{\underline{k}}$.

If, in lieu of solving for $g^{(2)}_{a_1 a_2}$ and removing the resulting secular behavior (which yields the kinetic equations for ψ_k), we examine the energy density of coulomb interaction associated with $g^{(2)}_{a_1 a_2}$ (see Appendix A), viz.

$$\Psi_{\underline{k}}^{(2)} = \frac{1}{K^4} \sum_{a_1, a_2} (4\pi n_{a_1} e_{a_1})(4\pi n_{a_2} e_{a_2}) \iint d\underline{v}_1 \, d\underline{v}_2 \; g^{(2)}_{a_1 a_2} , \qquad \text{III-64}$$

it can easily be demonstrated (see Appendix D and E) that the resulting expression is secular as t for large t. Demanding that the expression for this energy density be uniformly valid, we set the coefficient of t equal to zero and the kinetic equation for ψ_k on the ϵt scale results. We have for the kinetic equation for the waves (see Eq. 9, Appendix E)

$$\frac{\partial \Psi_{\underline{k}}}{\partial \epsilon t} = 2\gamma_{\underline{k}} \Psi_{\underline{k}} - 2\Psi_{\underline{k}} \, \mathrm{Re}\left(\sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2}{m_{a_1} K^2 \left.\frac{\partial \epsilon_{\underline{k}}}{\partial s}\right|_{s=-i\omega_{\underline{k}}}} \int \frac{d\underline{v}}{\Delta - i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}} \frac{\partial}{\partial \epsilon t} \frac{i\underline{k}\cdot \partial f^{(0)}_{a_1}/\partial \underline{v}}{\Delta - i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}} \right)$$

$$+ 2\Psi_{\underline{k}} \, \mathrm{Re}\Bigg(\frac{1}{\left.\frac{\partial \epsilon_{\underline{k}}}{\partial s}\right|_{s=-i\omega_{\underline{k}}}} \int d\underline{\ell}/(2\pi)^3 \; \Psi_{\underline{\ell}} \Bigg[\frac{\bar{\mu}(\underline{k},\omega_{\underline{k}}; -\underline{\ell}, -\omega_{\underline{\ell}}; \underline{k}-\underline{\ell}, \omega_{\underline{k}}-\omega_{\underline{\ell}})}{\epsilon(\underline{k}-\underline{\ell}, -i(\omega_{\underline{k}}-\omega_{\underline{\ell}}) + \Delta)}$$

$$\times \; \bar{\mu}(\underline{k}-\underline{\ell}, \omega_{\underline{k}}-\omega_{\underline{\ell}}; \underline{\ell}, \omega_{\underline{\ell}}; \underline{k}, \omega_{\underline{k}})$$

$$+ \sum_{a_1} \frac{e_{a_1}}{m_{a_1}} \frac{|\underline{k}-\underline{\ell}|^2}{K^2} \int d\underline{v}_1 \frac{1}{\Delta - i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}_1} \, i\underline{\ell}\cdot\frac{\partial}{\partial \underline{v}_1} \, \bar{\mu}^{a_1}_{\underline{v}_1}(\underline{k},\omega_{\underline{k}}; -\underline{\ell}, -\omega_{\underline{\ell}}; \underline{k}-\underline{\ell}, \omega_{\underline{k}}-\omega_{\underline{\ell}}) \Bigg] \Bigg)$$

$$+ \frac{\pi}{\left|\left.\frac{\partial \epsilon_{\underline{k}}}{\partial s}\right|_{s=-i\omega_{\underline{k}}}\right|^2} \int \frac{d\underline{\ell}}{(2\pi)^3} \Psi_{\underline{\ell}} \Psi_{\underline{k}-\underline{\ell}} \left| \bar{\mu}(\underline{k}-\underline{\ell}, \omega_{\underline{k}-\underline{\ell}}; \underline{\ell}, \omega_{\underline{\ell}}; \underline{k}, \omega_{\underline{k}}) \right|^2 \delta(\omega_{\underline{k}} - \omega_{\underline{\ell}} - \omega_{\underline{k}-\underline{\ell}}) ,$$

III-65

where
$$\gamma_{\underline{k}} = \mathrm{Re}\left(\sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2 / m_{a_1}}{K^2 \, \partial \epsilon_{\underline{k}}/\partial s |_{s=-i\omega_{\underline{k}}}} \int d\underline{v} \frac{i\underline{k}\cdot \partial f^{(1)}_{a_1}/\partial \underline{v}_1}{(\Delta - i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}_1)} \right) ,$$

and

$$\bar{\mu}(\underline{k}_1,\omega_1;\underline{k}_2,\omega_2;\underline{k}_3,\omega_3) = \sum_a \int d\underline{v}\ \bar{\mu}_{\underline{v}}^{a}(\underline{k}_1,\omega_1;\underline{k}_2,\omega_2;\underline{k}_3,\omega_3)$$

$$= \sum_a \frac{4\pi}{k_3^2}\frac{n_a e_a^3}{m_a^2}\int \frac{d\underline{v}}{(\Delta - i\omega_3 + i\underline{k}_3\cdot\underline{v})}\left\{ i\underline{k}_1\cdot\frac{\partial}{\partial\underline{v}}\ \frac{i\underline{k}_2\cdot\partial f_a^{(0)}/\partial\underline{v}}{\Delta - i\omega_2 + i\underline{k}_2\cdot\underline{v}} + (1\leftrightarrow 2)\right\}.$$

Using the net $g^{(2)}_{a_1 a_2}$ $(t \to \infty)$ from Eq. 12, Appendix E, and the first term of Eq. 21, Appendix D, in Eq. II-34 gives

$$\frac{\partial f_{a_1}^{(2)}}{\partial t} = 0 ,$$

and

$$\frac{\partial f_{a_1}^{(1)}}{\partial \epsilon t} + \frac{\partial f_{a_1}^{(0)}}{\partial \epsilon^2 t} = -\sum_{a_2=1}^{\alpha}(4\pi n_{a_2} e_{a_2})\, i\, \frac{e_{a_1}}{m_{a_1}}\int d\underline{k}\ \frac{\underline{k}}{k^2}\cdot\frac{\partial}{\partial \underline{v}_1}$$

$$\times \int d\underline{v}_2 \left\{ g_{a_2}(-\underline{k},\underline{v}_2)\ \xi_{\underline{k}}^{a_1}(\underline{v}_1) + \begin{pmatrix} \underline{k}\to -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix}\right.$$

$$\left. + \frac{1}{4}\int \frac{d\underline{\ell}}{(2\pi)^3}\ \psi_{\underline{\ell}}\,\psi_{\underline{k}-\underline{\ell}}\ \eta^{a_1 a_2}_{\underline{k}-\underline{\ell},\,\underline{\ell}} + \begin{pmatrix} \underline{k}\to -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix}\right\} \qquad \text{III-66}$$

where the notation is that of Appendices D and E. Equations III-65, III-66, and III-50 represent the final equations of the theory, excluding the effects of the collisional source term which, as previously discussed, may be ordered out of the problem for sufficiently small ϵ_p .

CHAPTER IV

PROPERTIES OF THE FINAL EQUATIONS

Equations III-50, III-65 and III-66 represent the generalization of the usual quasi-linear results to the case of a multicomponent plasma including the possibility of mode coupling effects encountered through terms involving $\delta(\omega_k - \omega_\ell - \omega_{k-\ell})$. These equations have the desirable property that certain conservation laws are upheld.

a) Total Probability Conservation

Equations III-50 and III-66 conserve probability for each species since

$$\frac{\partial}{\partial \epsilon t} \int f_{a_1}^{(0)} d\underline{v}_1 = 0$$

and

$$\frac{\partial}{\partial \epsilon t} \int f_{a_1}^{(1)} d\underline{v}_1 + \frac{\partial}{\partial \epsilon^2 t} \int f_{a_1}^{(0)} d\underline{v}_1 = 0$$

trivially upon integration by parts.

b) Momentum Conservation

From Eq. III-50 we note that

$$\begin{aligned}
\frac{\partial}{\partial \epsilon t} \underline{P}^{(0)} &= \sum_{a_1=1}^{\alpha} n_{a_1} m_{a_1} \frac{\partial}{\partial \epsilon t} \int d\underline{v}_1 \, \underline{v}_1 \, f_{a_1}^{(0)} \\
&= \sum_{a_1} \frac{\pi n_{a_1} e_{a_1}^2}{m_{a_1}} \iint d\underline{k}\, d\underline{v}_1 \, \Psi_{\underline{k}} \, \underline{k}\, \underline{k} \cdot \frac{\partial f_{a_1}^{(0)}}{\partial \underline{v}_1} \delta(\omega_{\underline{k}} - \underline{k} \cdot \underline{v}_1) \\
&= -\int d\underline{k} \, \underline{k} \, \Psi_{\underline{k}} \left(\sum_{a_1} \frac{\pi n_{a_1} e_{a_1}^2}{m_{a_1}} \int d\underline{v}_1 \, \underline{k} \cdot \frac{\partial f_{a_1}^{(0)}}{\partial \underline{v}_1} \delta(\omega_{\underline{k}} - \underline{k} \cdot \underline{v}_1) \right)
\end{aligned}$$

$\equiv 0$, since (for range of $\underline{k}$ we consider)

$$\sum_{a_1} \frac{\pi n_{a_1} e_{a_1}^2}{m_{a_1}} \int d\underline{v}_1 \, \underline{k} \cdot \frac{\partial f_{a_1}^{(0)}}{\partial \underline{v}_1} \delta(\omega_{\underline{k}} - \underline{k} \cdot \underline{v}_1) = 0$$

is a consequence of $\epsilon(\underline{k}, -i\omega_{\underline{k}} + \Delta) = 0$.

Similarly from Eq. III-66

$$\frac{\partial}{\partial \epsilon t} \underline{P}^{(1)} + \frac{\partial}{\partial \epsilon^2 t} \underline{P}^{(0)} \equiv 0 \, .$$

This follows directly from noting that

$$\sum_{a_1=1}^{\alpha} n_{a_1} m_{a_1} \int d\underline{v}_1 \, \underline{v}_1 \times \left(\text{the right-hand side of Eq. III-66} \right)$$

reduces to an integral of the form

$$\int d\underline{k} \, \underline{k} \times (\text{symmetric function of } \underline{k}) \equiv 0 .$$

c) Energy Conservation

$$\frac{\partial}{\partial \epsilon t} T^{(0)} = \sum_{a_1} n_{a_1} m_{a_1} \frac{\partial}{\partial \epsilon t} \int d\underline{v}_1 \, \frac{v_1^2}{2} f_{a_1}^{(0)} = 0$$

follows trivially from Eq. III-50 upon integrating by parts and using $\omega_{-\underline{k}} = -\omega_{\underline{k}}$. Energy conservation in the case of Eq. III-66 is somewhat more tedious, but operating upon Eq. 66 with $\sum_{a_1} n_{a_1} m_{a_1} \int d\underline{v}_1 \, \frac{v_1^2}{2}$, integrating by parts with respect to $\underline{v}_1$, using Eq. III-65 and the explicit form of the responses $\bar{\mu}$ gives

$$\frac{\partial}{\partial \epsilon t} T^{(1)} + \frac{\partial}{\partial \epsilon^2 t} T^{(0)} = -\frac{1}{8\pi} \frac{\partial}{\partial \epsilon t} \int d\underline{k} \, k^2 \Psi_{\underline{k}} \, . \qquad \text{IV-1}$$

Use is also made of writing $i\underline{k}\cdot\underline{v}_1 = i\underline{k}\cdot\underline{v}_1 - i\omega_k + i\omega_k$ in the integrals to be considered. IV-1 is just a statement of the conservation of particle kinetic energy + coulomb interaction energy, $\frac{k^2 \Psi_k}{8\pi}$. This brings us to an important point which we have not commented upon, that is, the sign of the spectrum $\Psi_{\underline{k}}$.

d) Sign of the Spectrum of Coulomb Interaction Density

Close examination of expression III-47 for $\Psi_{\underline{k}}$ in terms of the initial value $g^{(1)}(\underline{k}, \underline{v}_1, \underline{v}_2, 0, \epsilon t)$ shows that

$$\Psi_{\underline{k}} = \Psi_{-\underline{k}} = \Psi_{\underline{k}}^{*} .$$

This follows changing $\underline{k}$ to $-\underline{k}$ and using the reality condition on $g^{(1)}_{a_1 a_2}$. This is all we can say concerning the spectrum $\psi_{\underline{k}}$ from the explicit expression for it, i.e., that it is real and symmetric in $\underline{k}$. We can not say it is positive definite. Specific use of the assumption that it is positive definite has been used in stabilization arguments in the literature,[1] however, it is clear that an auxiliary argument must be given to justify this. Since Eq. III-65 is a kinetic equation for the evolution of $\psi_{\underline{k}}$ we can hope to make the following statement. If the spectrum $\psi_{\underline{k}}$ is initially positive it remains so. Let us assume that the spectrum is initially positive and that it first turns negative for $\underline{k} = \underline{k}_o$. Equation III-65 then yields for this $\underline{k}_o$,

$$\frac{\partial}{\partial \epsilon t} \psi_{\underline{k}_o} = \pi \int \frac{d\underline{\ell}}{(2\pi)^3} \frac{\psi_{\underline{\ell}} \psi_{\underline{k}_o - \underline{\ell}}\, \delta(\omega_{\underline{k}_o} - \omega_{\underline{\ell}} - \omega_{\underline{k}_o - \underline{\ell}}) \left| \bar{\mu}(\underline{k}_o - \underline{\ell}, \omega_{\underline{k}_o - \underline{\ell}}; \underline{\ell}, \omega_{\underline{\ell}}; \underline{k}_o, \omega_{\underline{k}_o}) \right|^2}{\left| \partial \epsilon_{\underline{k}} / \partial s \right|^2_{s = -i\omega_{\underline{k}}}} \qquad \text{IV-2}$$

i.e., if the condition

$$\omega_{\underline{k}_o} - \omega_{\underline{\ell}} = \omega_{\underline{k}_o - \underline{\ell}} \qquad \text{IV-3}$$

can be satisfied, then at the instant $\psi_{\underline{k}_o}$ is passing through zero,

$$\frac{\partial}{\partial \epsilon t} \psi_{\underline{k}_o} > 0 . \qquad \text{IV-4}$$

That is to say, the spectrum does not turn negative if it is initially positive. The reason for this has been the positive-definite nature of the mode-coupling driving term in the kinetic equation for the waves. In situations where the mode coupling terms are absent (when IV-3 cannot be satisfied) and thus $\frac{\partial \psi_{\underline{k}_o}}{\partial \epsilon t} = 0$, one could proceed by examining the signs of higher-order time derivatives of $\psi_{\underline{k}_o}$, i.e., $\frac{\partial^2 \psi_{\underline{k}_o}}{\partial \epsilon t^2}$, and so on. However, in lieu of following this procedure, it is convenient to determine the effect of the collisional source term $S_{a_1 a_2}$ in Eq. II-36 on the kinetic equation for the waves, III-65.

We anticipate that this will add a positive definite inhomogeneous source term on the right-hand side of III-65 which will ensure that the spectrum $\psi_{\underline{k}}$ remains positive if initially so, whether or not IV-3 can be satisfied. The approach we take is similar to that in Appendix D and E, namely, we calculate the contribution of the source term $S_{a_1 a_2}$, which drives $g^{(2)}_{a_1 a_2}$ in Eq. II-36, to the second-order energy density of coulomb interaction

$$\frac{1}{K^4}\sum_{a_1,a_2}(4\pi n_{a_1}e_{a_1})(4\pi n_{a_2}e_{a_2})\iint d\underline{v}_1\, d\underline{v}_2\; g^{(2)}_{a_1 a_2}\,. \qquad \text{IV-5}$$

This will be secular as t for large t and hence add a driving term to the kinetic equation for $\psi_{\underline{k}}$ (Eq. III-65) when we ask that the total second-order energy density be given by a uniformly valid expression. From Eq. II-36, the contribution to $g^{(2)}_{a_1 a_2}$ from the driving term $S_{a_1 a_2}$ in terms of Laplace variables is

$$g^{(2)}_{a_1 a_2} = \frac{1}{s\,(s + L^{(0)}_{a_1}(\underline{k},\underline{v}_1) + L^{(0)}_{a_2}(-\underline{k},\underline{v}_2))}\;\frac{4\pi e_{a_1}e_{a_2}}{K^2\, m_{a_1}}\, i\underline{k}\cdot\frac{\partial f^{(0)}_{a_1}}{\partial \underline{v}_1}\, f^{(0)}_{a_2} + \left(\begin{matrix}\underline{k}\to-\underline{k}\\ 1\leftrightarrow 2\end{matrix}\right). \qquad \text{IV-6}$$

Then IV-5 becomes

$$\frac{4\pi}{K^4}\int_{C_1}\!\!\int_{C_2}\frac{ds_1\,ds_2/(2\pi i)^2}{s(s-s_1-s_2)}\,\frac{1}{\epsilon(\underline{k},s_1)}\,\frac{1}{\epsilon(-\underline{k},s_2)}\sum_{a_1}\frac{4\pi n_{a_1}e^2_{a_1}}{m_{a_1}K^2}\int\frac{i\underline{k}\cdot\partial f^{(0)}_{a_1}/\partial\underline{v}_1}{s_1+i\underline{k}\cdot\underline{v}_1}\sum_{a_2}4\pi\int\frac{n_{a_2}e^2_{a_2}\, f^{(0)}_{a_2}\, d\underline{v}_2}{s_2 - i\underline{k}\cdot\underline{v}_2}$$
$$+\,(\underline{k}\to-\underline{k})\,, \qquad \text{IV-7}$$

where we have used the identity

$$\sum_a(4\pi n_a e_a)\int d\underline{v}\,\frac{1}{s+L^{(0)}_a(\underline{k},\underline{v})} \equiv \frac{1}{\epsilon(\underline{k},s)}\sum_a(4\pi n_a e_a)\int\frac{d\underline{v}}{s+i\underline{k}\cdot\underline{v}}\,. \qquad \text{IV-8}$$

Effectively the zero of $\epsilon(\underline{k},s_1)$ at $s_1=-i\omega_{\underline{k}}$ and $\epsilon(-\underline{k},s_2)$ at $s_2=i\omega_{\underline{k}}$, lead to a $1/s^2$ behavior in IV-7 which thus gives a secularity proportional to t. We examine this in Appendix F. The result for large t for the inversion of Eq. IV-7 is

$$t\,\delta_{\underline{k}}\,, \qquad \text{IV-9}$$

where

$$\mathcal{S}_{\underline{k}} = \frac{1}{\left|\frac{\partial \varepsilon_k}{\partial s}\right|^2_{s=-i\omega_k}} \frac{4\pi}{K^4} \sum_{a_2} 4\pi n_{a_2} e_{a_2}^2 \int \frac{f_{a_2}^{(0)}(\underline{v})\, d\underline{v}}{(\Delta - i\omega_k + i\underline{k}\cdot\underline{v})} + (\underline{k} \to -\underline{k}) \qquad \text{IV-10}$$

$$= \frac{1}{\left|\frac{\partial \varepsilon_k}{\partial s}\right|^2_{s=-i\omega_k}} \sum \frac{(2\pi)(4\pi)^2 n_a e_a^2}{K^4} \int f_a^{(0)}(\underline{v})\, \delta(\omega_k - \underline{k}\cdot\underline{v})\, d\underline{v} .$$

Asking that the net second-order energy density be uniformly valid with the inclusion of Eq. IV-9 just adds to the right-hand side of Eq. III-65 the inhomogeneous source term $\mathcal{S}_{\underline{k}}$ given in Eq. IV-10. This term is positive definite and ensures together with the mode coupling term in Eq. IV-2 that the spectrum $\Psi_{\underline{k}}$ does not turn negative if initially positive. Moreover if the spectrum $\Psi_{\underline{k}}$ is initially zero everywhere, the source term $\mathcal{S}_{\underline{k}}$ allows the spectrum to be driven into the system, and in fact $\Psi_{\underline{k}}$ will be positive definite since $\mathcal{S}_{\underline{k}} > 0$. $S_{a_1 a_2}$ will also lead to a nonsecular, $t \to \infty$, contribution to $g^{(2)}_{a_1 a_2}$ and thus give a collisional driving term in Eq. III-66 for the first distribution function. This has been calculated in Reference 1 for a gas of electrons in a fixed background of neutralizing ions, and we do not pursue it further. However for the verification of any conservation laws it is a necessary inclusion when $\mathcal{S}_{\underline{k}}$ is included in the kinetic equation for the waves, Eq. III-65.

e) Comments on Eq. III-65 and Kadomtsev's Kinetic Equation for the Waves [10]

Eq. III-65 and III-66 represent a generalization of the usual quasi-linear results to a multicomponent plasma including the effect of wave-wave interactions through the appearance of terms containing $\delta(\omega_k - \omega_\ell - \omega_{k-\ell})$. We note that the driving terms in the equation for the waves, Eq. III-65 are identical to those given in Kadomstev's "Plasma Turbulence", with the exception of the second term on the right hand side, which is absent in his treatment. We can best understand this discrepancy by familiarizing ourselves with the significance of this term. It represents a driving term in the kinetic equation for the spectrum ψ_k resulting from the variation of ω_k on the ϵt time scale. The dispersion relation

$$\epsilon(\underline{k}, -i\omega_{\underline{k}} + \Delta) = 0$$

is adiabatic. It is an integral equation interrelating $f^{(o)}$ and $\omega_{\underline{k}}$. Although $f^{(o)}$ does not vary on the t scale, a variation in ϵt is allowed through Eq. III-50, hence ω_k may vary on the ϵt scale. We also observe that the second term in Eq. III-65 is manifestly nonlinear in ψ. This follows since the variation of $f^{(o)}$ on the ϵt scale in Eq. III-50 depends linearly on ψ. In the interesting case of a one-dimensional electron plasma in a fixed background of positive ions this term is absent in the kinetic equation for the waves. In this case the dispersion relation implies

$$\frac{\partial f^{(o)}}{\partial v}\left(\frac{\omega_k}{k}\right) = 0 \quad ,$$

for k in the marginally stable region. Equation III-50 then gives $\frac{\partial f^{(o)}}{\partial \epsilon t} = 0$. Thus, since neither $f^{(o)}$ nor ω_k vary on the ϵt scale, the above statement follows.

It is not apparent from Kadomtsev's treatment that one should expect to see the effect included in the second term of Eq. III-65 using his analysis. We feel that the presence of such a term is in fact quite plausible. The important feature is that both theories lead to similar kinetic equations for the waves although the starting points are quite dissimilar. We have been looking at turbulence in a spatially homogeneous ensemble through the B-B-G-K-Y hierarchy of equations. Kadomtsev's approach has been one of studying homogeneous turbulence in a Vlasov plasma, as in the approach in Ref. 7. In Chapter VI we demonstrate that these two approaches are in fact describing the same phenomena and should lead to similar results.

f) <u>One-Dimensional Plasma</u>

We turn to the problem of a one-dimensional electron gas in a fixed background of neutralizing ions. As previously stated, in this case $\frac{\partial f_0}{\partial v} = 0$ in the region of marginally stable phase velocities and $f^{(0)}$ does not vary on the ϵt scale. In the various responses $\bar{\mu}$, the contribution from the $\delta(\omega_k - kv)$ portion of the linear resonance terms,

$$\frac{1}{(\omega_k - kv + i\Delta)},$$

is absent as $\frac{\partial f_0}{\partial v} = 0$ for those $v = \frac{\omega_k}{k}$. Since $v = \frac{\omega_k - \omega_\ell}{k - \ell}$ may well be outside the region of marginal stability, the $\delta(\omega_k - \omega_\ell - (k-\ell)v)$ terms must be included in the nonlinear resonances,

$$\frac{1}{(\omega_k - \omega_\ell - (k-\ell)v + i\Delta)}.$$

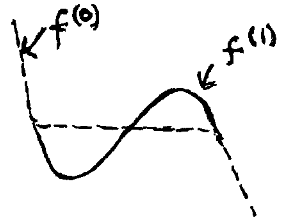
Frieman and Rutherford [1] have calculated in detail the various coupling coefficients associated with the equations for $f^{(1)}$ and the energy spectrum under the assumption

$$\omega_k - \omega_\ell - \omega_{k-\ell} \neq 0.$$

In reference to the equation for the first distribution they obtain for v in the range of marginally stable phase velocities

$$\frac{\partial f^{(1)}}{\partial \epsilon t} = \left(\frac{e}{m}\right)^2 \pi \int dk\, \Psi_k\, k \frac{\partial}{\partial v}\left(\delta(\omega_k - kv)\, k \frac{\partial f^{(1)}}{\partial v}\right). \qquad \text{IV-11}$$

The inclusion of the wave-wave terms involving $\delta(\omega_k - \omega_\ell - \omega_{k-\ell})$ does not alter Eq. IV-11 for v in the marginally stable region. This is a trivial observation following an examination of the explicit form of Eq. III-66 and utilizing $\partial f^{(0)}/\partial v = 0$ in this region. Keeping in mind our conclusions drawn concerning the sign of the spectrum Ψ_k, namely, that if initially positive it never turns negative, a quasi- H theorem may be proven in reference to Eq. IV-11 . With $f^{(1)}$ of the form to yield an instability, i.e., zero at the end points of plateau in $f^{(0)}$ and a region of positive slope in between ,



we multiply Eq. IV-11 by $f^{(1)}$ and integrate over v_1 between end points yielding

$$\frac{\partial}{\partial \epsilon t} \int \frac{f^{(1)2}}{2} dv = -\pi \left(\frac{e}{m}\right)^2 \int dk\, \Psi_k \int dv\, \delta(\omega_k - kv) \left(k \frac{\partial f^{(1)}}{\partial v}\right)^2. \qquad \text{IV-12}$$

However by hypothesis $\Psi_k > 0$ initially and hence remains so; Eq. IV-12 thus implies as $\epsilon t \to \infty$, $\frac{\partial f^{(1)}}{\partial v} \to 0$ for those $v = \omega_k / k$, and the stabilization argument goes through as in Reference 1.

We turn to an examination of the kinetic equation for the waves, Eq. III-66. As is prevalent in the literature we introduce the concept of quasi-particles.[7] In problems where the amplitude of the electric field $\hat{E}_k$ is subject to slow variations in time the significant energy density to consider is

$$\mathcal{E}_k = \frac{|\hat{E}_k|^2}{8\pi} \omega_k \left. \frac{\partial \epsilon(k,\omega)}{\partial \omega} \right|_{\omega=\omega_k} . \qquad \text{IV-13}$$

This represents the $\frac{|\hat{E}_k|^2}{8\pi}$ energy of the electrostatic field plus the polarization energy due to the interaction of the particles with the field. In Appendix G, we give a simple argument based on a linearized Vlasov model which indicates that Eq. IV-13 is the pertinent energy density. The number of quasi-particles in the k-th mode is defined to be

$$n_k = \frac{\mathcal{E}_k}{\omega_k} , \qquad \text{IV-14}$$

i.e., the effective wave energy (field plus polarization energies) in the mode divided by $\omega_{\underline{k}}$. This is an especially useful concept to use in reference to the mode coupling terms in Eq. III-65 (those involving $\delta(\omega_k - \omega_\ell - \omega_{k-\ell})$) since these terms take on a relatively simple form. Although the relation $\omega_k - \omega_\ell - \omega_{k-\ell} \sim 0$ is impossible to satisfy in the long-wave length limit ($\omega_k \sim \omega_p$), we do not preclude the possibility that it could be satisfied in situations where the instability region is closer to the central portion of the main distribution. In any case, it is of considerable interest to determine whether or not these mode coupling terms conserve quasi-energy $n_k \omega_k$ and quasi-momentum $n_k\, k$. It should perhaps be surprising if they did not since the microscopic phenomenon is manifestly conservative in energy and momentum, i.e.,

$$\omega_{k_1} + \omega_{k_2} = \omega_{k_3} ,$$
$$k_1 + k_2 = k_3 .$$

In reference to Eq. III-66, we introduce the number of quasi-particles,

$$n_k \omega_k = \frac{k^2}{8\pi} \Psi_k\, \omega_k \left.\frac{\partial \mathcal{E}(k,\omega)}{\partial \omega}\right|_{\omega_k} . \qquad \text{IV-15}$$

It is also convenient to rescale the responses $\bar{\mu}$ (now defined for a single species of electrons) as

$$\bar{M}(k_1,\omega_1; k_2,\omega_2; k_3,\omega_3) = \sqrt{8\pi}\, k_3^2\, \bar{\mu}(k_1,\omega_1; k_2,\omega_2; k_3,\omega_3) , \qquad \text{IV-16}$$

where $\bar{\mu}$ is defined in Eq. 6, Appendix E. We observe that

$$\bar{M}(k_1,\omega_1; k_2,\omega_2; k_3,\omega_3) = \bar{M}(k_2,\omega_2; k_1,\omega_1; k_3,\omega_3) , \qquad \text{IV-17}$$

and

$$\bar{M}^*(k_1,\omega_1; k_2,\omega_2; k_3,\omega_3) = \bar{M}(-k_1,-\omega_1; -k_2,-\omega_2; -k_3,-\omega_3) . \qquad \text{IV-18}$$

A very useful result following upon integration by parts is that

IV-19

$$\bar{M}(k,\omega_k; -\ell,-\omega_\ell; k-\ell, \omega_k-\omega_\ell)$$
$$= \bar{M}(k-\ell, \omega_k-\omega_\ell; \ell,\omega_\ell; k,\omega_k) .$$

We also introduce the notation

$$\epsilon_k' = -\frac{\omega_p^2}{k^2}\int \frac{k\, \partial f_0/\partial v}{(\omega_k - kv)^2}\, dv . \tag{IV-20}$$

With Eqs. IV-15, -16 , and -20, Eq. III-65 can be rewritten as

$$\frac{\partial n_k}{\partial \epsilon t} = 2\gamma_k n_k + R(n) + W(n), \tag{IV-21}$$

where R(n) describes the nonlinear wave-particle resonance phenomena stemming from those $v = (\omega_k - \omega_\ell)/(k - \ell)$ and is given by

$$R(n) = \int \frac{d\ell}{(2\pi)} n_k n_\ell R_{k\ell}, \tag{IV-22}$$

where $R_{k\ell} = 2\,\mathrm{Im}\left\{ \frac{P\ \bar{M}(k,\omega_k; -\ell, -\omega_\ell; k-\ell, \omega_k-\omega_\ell)\bar{M}(k-\ell, \omega_{k-\ell}; \ell, \omega_\ell; k, \omega_k)}{\epsilon(k-\ell, -i(\omega_k-\omega_\ell)) \quad k^2\epsilon_k'\ \ell^2\epsilon_\ell'\ (k-\ell)^2} \right.$

$$\left. - \frac{e}{m}\frac{1}{k^2\epsilon_k'\ell^2\epsilon_\ell'}\int \frac{dv}{\omega_k - kv}\,\ell\,\frac{\partial}{\partial v}\,\bar{M}_v(k,\omega_k; -\ell, -\omega_\ell; k-\ell, \omega_k-\omega_\ell)\right\}.$$

The expression W(n) describes the wave-wave interaction phenomena for $\omega_k - \omega_\ell = \omega_{k-\ell}$ and is given by

$$W(n) = \pi\int \frac{d\ell}{(2\pi)} \frac{\delta(\omega_k - \omega_\ell - \omega_{k-\ell})}{(k^2\epsilon_k'\ell^2\epsilon_\ell'(k-\ell)^2\epsilon_{k-\ell}')}$$

$$\times \left\{ n_\ell n_{k-\ell} |\bar{M}(k-\ell, \omega_{k-\ell}; \ell, \omega_\ell; k, \omega_k)|^2 \right. \tag{IV-23}$$

$$\left. - n_\ell n_k\, 2\,\mathrm{Re}\left(\bar{M}(k,\omega_k; -\ell, -\omega_\ell; k-\ell, \omega_{k-\ell})\bar{M}(k-\ell, \omega_{k-\ell}; \ell, \omega_\ell; k, \omega_k)\right)\right\}.$$

Using Eq. IV-15, Eq. IV-17, Eq. IV-18 and -19 together with $\omega_{k-\ell} = \omega_k - \omega_\ell$, Eq. IV-23 can be rewritten as

$$W(n) = \pi\int \frac{d\ell}{(2\pi)} \frac{\delta(\omega_k - \omega_\ell - \omega_{k-\ell})}{(k^2\epsilon_k'\ell^2\epsilon_\ell'(k-\ell)^2\epsilon_{k-\ell}')} \times |\bar{M}(k-\ell, \omega_{k-\ell}; \ell, \omega_\ell; k, \omega_k)|^2$$

$$\times\,\delta(\omega_k - \omega_\ell - \omega_{k-\ell}) \times (n_\ell n_{k-\ell} - n_k n_\ell - n_k n_{k-\ell}), \tag{IV-24}$$

$$\equiv \int \frac{d\ell}{(2\pi)}\,\delta(\omega_k - \omega_\ell - \omega_{k-\ell})\, W_{k\ell}\,(n_\ell n_{k-\ell} - n_k n_\ell - n_k n_{k-\ell}).$$

Eq. IV-24 conserves quasi-energy, $n_k \omega_k$, since

$$\int dk\, \omega_k \left(\frac{\partial n_k}{\partial \epsilon t} \right)_W$$
$$= \pi \iint \frac{d\ell\, dk}{(2\pi)}\, \delta(\omega_k - \omega_\ell - \omega_{k-\ell})\, \left| \bar{M}(k-\ell, \omega_{k-\ell}; \ell, \omega_\ell; k, \omega_k) \right|^2$$
$$\times \Big(n_\ell (n_{k-\ell}\omega_{k-\ell} - n_k \omega_k) + n_{k-\ell}(n_\ell \omega_\ell - n_k \omega_k) \Big). \qquad \text{IV-25}$$

Upon letting $\ell \to k - \ell$ before doing the k integration, and using Eq. IV-17, this can be rewritten as

$$= 2\pi \iint \frac{d\ell\, dk}{(2\pi)}\, \delta(\omega_k - \omega_\ell - \omega_{k-\ell}) \left| \bar{M}(k-\ell, \omega_{k-\ell}; \ell, \omega_\ell; k, \omega_k) \right|^2$$
$$\times\, n_{k-\ell}\, (n_\ell \omega_\ell - n_k \omega_k) \quad ,$$

which is antisymmetric under interchange of k and ℓ upon using Eq. IV-17 -18, and -19, and thus gives zero. Similarly quasi-momentum $n_k k$ is conserved by $W(n)$.

In reference to the effects of mode-coupling, the reader should consult Chap. V where we consider a simple model equation exhibiting solely wave-wave interactions. This example results in a kinetic equation for the wave energy density with driving terms identical in form to the mode coupling terms in III-65, and is discussed in considerable detail in Chap. V. For example, the observation that $n_k = T/\omega_k$ (V-44) is a stationary solution for the number of quasi-particles, is also true in relation to the equation $\frac{\partial n_k}{\partial \epsilon t} = W(n)$, discussed above.

As discussed earlier, $f^{(1)}$ flattens for large ϵt and the growth rate γ_k tends to zero as $\epsilon t \to \infty$. The asymptotic form of the spectrum $\psi_{\underline{k}}$ (or number of quasi-particles n_k in Eq. IV-21), however, depends upon the details of wave-particle coupling coefficent, $R_{k\ell}$, and the mode-coupling coefficient, $W_{k\ell}$. These must be evaluated in the context of the particular physical situation being studied as they depend on ω_k and the details of $f^{(o)}$. This could involve subsidiary, long-($\lambda > \lambda_D$) or short-($\lambda < \lambda_D$) wavelength expansions. It is of interest to note that the various terms in Eq. IV-21 have simple physical interpretations in terms of the emission or absorption of plasmons[19] (quasi-particles). The driving term $2\gamma_k n_k$ (for $\gamma_k > 0$) corresponds to stimulated plasmon emission arising from wave-particle interactions giving resonant behavior for $\omega_k - k v \backsim 0$. The quantity $R(n)$ corresponds to plasmon emission, or absorption due to wave-particle resonant behavior for $\omega_k - \omega_\ell - (k - \ell) v \backsim 0$. The emission or absorption of plasmons caused by wave-wave interactions is described by $W(n)$. If the effect of the source term $\mathcal{S}_k$ (given in Eq. IV-10) had been included in Eq. IV-21, this would correspond to the spontaneous emission of plasmons resulting from single particle encounters.

GENERAL COMMENTS

In this section we wish to indicate at least some of the conditions for the validity of Chapter IV. This also has a direct bearing upon much of the quasilinear theory of weak turbulence prevalent in the literature. We divide the discussion into three parts: the first dealing with the streaming terms $e^{-i\underline{k}\cdot\underline{v}t}$, the second with conditions on the validity of the inclusion of the mode coupling terms encountered through $\delta(\omega_k - \omega_\ell - \omega_{k-\ell})$, and thirdly some comments on continuous and discrete representations.

g) Free-Streaming Terms

Here, as elsewhere in the literature,[1, 3-11] we have consistently neglected poles associated with free-streaming motion, i. e., poles at $s = -i\underline{k}\cdot\underline{v}$ leading to $e^{-i\underline{k}\cdot\underline{v}t}$ behavior. The general philosophy has been that such terms tend to mix to zero in velocity integrals. Let us examine a typical term we have not included in reference to

$$i\omega_p^2 \int d\underline{k}\, \frac{\underline{k}}{k^2} \cdot \iint \frac{\partial \oplus}{\partial \underline{v}_1}\, g_{\underline{k}}(\underline{v}_1, \underline{v}_2, t)\, d\underline{v}_1\, d\underline{v}_2 \; . \qquad \text{IV-26}$$

Without loss of generality we work with a single species of electrons. It is clear (see also Appendix C_1) that such a term is of the form

$$i\omega_p^2 \int d\underline{k}\; \underline{k} \cdot \int \frac{\partial}{\partial \underline{v}_1} \oplus(\underline{v}_1)\, f(\underline{k}, \underline{v}_1)\, e^{i(\omega_{\underline{k}} - \underline{k}\cdot\underline{v}_1)t}\, d\underline{v}_1 \, , \qquad \text{IV-27}$$

where we have assumed that $\oplus$ and $g^{(1)}_{\underline{k}}(\underline{v}_1, \underline{v}_2, 0, \epsilon t \cdots)$ are analytic in sufficiently broad strips containing $\underline{v}_1$, $\underline{v}_2$ real. The $e^{i\omega_{\underline{k}} t}$ arises from $\int d\,\underline{v}_2\, g_{\underline{k}}^{(1)}$, and the $e^{-i\underline{k}\cdot\underline{v}\,t}$ factor comes from the $\frac{1}{s_1 + i\underline{k}\cdot\underline{v}}$ pole. We estimate the rate of decay of Eq. IV-27 in the one-dimensional problem (this usually leads to a pessimistic estimate since in the three-dimensional problem there are two extra degrees of freedom for mixing to occur). For simplicity it is convenient to assume that $\frac{\partial \oplus}{\partial v_1} f(k_1\, v_1)$ is factorable,

$$\frac{\partial \oplus}{\partial v_1} f(k, v_1) = f(v_1)\, g(k) .$$

Defining $G(k) = -ik\, g(k)$ and also noting that

$$G(-k) = G(k)^* \quad , \qquad \text{IV-28}$$

from the reality condition on $g^{(1)}(\underline{x}\, ,\, \underline{v}_1\, ,\, \underline{v}_2)$, we have from Eq. IV-27

$$I = \omega_p^2 \int dk\, G(k)\; e^{i\omega_k t} \left(\int dv_1\, f(v_1)\, e^{-ikv_1 t} \right) \quad .$$

We now illustrate how rapidly I decays for two particular choices of $f(v_1)$.

1. If $f(v_1)$ were Maxwellian about some v_o

then
$$I_1 = \omega_p^2 \int dk\, G(k)\, e^{i\omega_k t} \left(e^{-ikv_o t}\, e^{-\frac{k^2\Delta^2 t^2}{2}} A \sqrt{\pi\Delta^2} \right) \qquad \text{IV-29}$$

as
$$f(v_1) \quad A\; e^{-(v_1 - v_o)^2/2\Delta^2} \quad .$$

2. If $f(v_1)$ were a resonant function about v_o , i.e.,

$$f = \frac{A}{(v_1 - v_o)^2 + \Delta^2} \quad ,$$

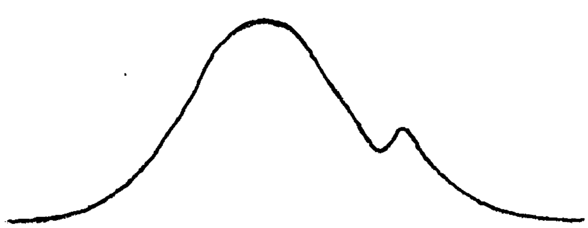
then

$$I_2 = \omega_p^2 \int dk\, G(k)\; e^{i\omega_k t} \left(e^{-ikv_o t}\; e^{-|k|\Delta\, t}\; A\, \frac{\pi}{\Delta} \right) \quad . \qquad \text{IV-30}$$

In both cases the integrand of the resulting k-integral damps rapidly as has been assumed throughout the analysis. We may rewrite,using Eq. IV-28,

$$\begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \omega_p^2 2\,\mathrm{Re} \int_{\kappa \geq 0} d\kappa\, G(\kappa)\, e^{i(\omega_\kappa - \kappa v_0)t} \begin{Bmatrix} e^{-\frac{\kappa^2 \Delta^2 t^2}{2}} A \sqrt{\pi \Delta^2} \\ e^{-|\kappa| \Delta t} A\, \pi/\Delta \end{Bmatrix} \qquad \text{IV-31}$$

We preface estimates of the rate at which I_1 and I_2 damp by a discussion of the limits of κ-integration. Throughout this theory we have consistently neglected phenomena associated with κ outside the region of slight instability under the philosophy that this corresponds to the region of Landau damping and after sufficiently long time leads to negligible contribution. Since in a long-wavelength expansion the Landau damping increment itself is exponentially small, it is preferable to the validity of the theory to avoid situations where the small κ modes are strongly excited. This would be the case if the region of small instability is not far out in the distribution tail, as in the following diagram :



In the long-wave length limit particle-particle encounters are the effective mechanism for damping, which is out of the scope of this theory. Actually for long (or short) wave lengths the hierarchy equations require a re-ordering and are not those being solved for here. The importance of particle encounters for the damping of long-wave length phenomena can be seen most simply in a somewhat simplified model. In the linearized Vlasov problem if we add to the right-hand side a small collision term of the Krook form, $-\nu f^{(1)}$, where ν is assumed constant, then for a resonant background distribution function,

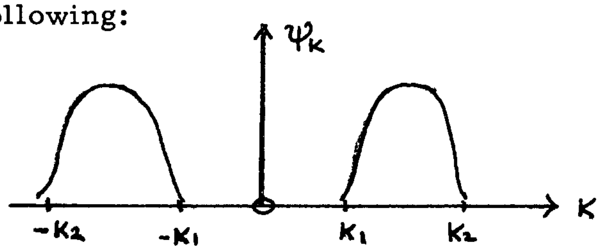
$$f_0 = \frac{b}{\pi(v^2+b^2)},$$ a Landau analysis yields

$$E_k \sim \hat{E}_k e^{-i\omega_p t} e^{-\gamma t},$$

where $$\gamma = |k| b + \nu.$$

As $k \to 0$, the damping is predominantly from ν, the effective collision frequency.

In any case we limit ourselves to situations where the k dependence of functions is vanishingly small outside the intervals $[k_1, k_2]$ and $[-k_2, -k_1]$. For example the spectrum of energy density $\psi_k \; (= \psi_{-k} = \psi_k^*)$ may resemble the following:



Upper bounds on the rate of decay of I_1(or I_2) are then simply obtained. We assume $|G|$ takes a maximum value G_0 on $[k_1, k_2]$, then

$$\left\{ \begin{matrix} |I_1| \\ |I_2| \end{matrix} \right\} \leq 2\omega_p^2 G_0 \int_{k_1}^{k_2} \left\{ \begin{matrix} e^{-k^2\Delta^2 t^2/2} A\sqrt{\pi\Delta^2} \\ e^{-|k|\Delta t} A\pi/\Delta \end{matrix} \right\} dk \qquad \text{IV-32}$$

$$= \omega_p^2 2 G_0 \left\{ \begin{matrix} \frac{A\pi}{\sqrt{2}} \cdot \frac{1}{t} \left(\mathrm{erf}\left(\frac{k_2 \Delta t}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{k_1 \Delta t}{\sqrt{2}}\right)\right) \\ \frac{A\pi}{\Delta^2} \cdot \frac{1}{t} \left(e^{-k_1 \Delta t} - e^{-k_2 \Delta t}\right) \end{matrix} \right\},$$

$\longrightarrow 0$ in a time t such that

$$k_0 \Delta t \gg 1, \qquad \text{IV-33}$$

where K_0 is typical of the range $[K_1, K_2]$. In actual fact for the long times the slowest decay in IV-33 is from the long-wave length, K_1 - limit of integration and IV-32 should really read

$$K_1 \Delta t \gg 1 . \qquad \text{IV-34}$$

The advantage of such exponential decay is clear. In higher order (e.g., the calculation of $g^{(2)}$) where we have consistently neglected e^{-ikvt} streaming terms, the effect of including such (e.g., in the calculation of $H^{(2)}$ and hence $g^{(2)}$) is to yield $t^n e^{-ikvt}$ behaviour, where n depends on the number of velocity derivatives operating on the free-streaming terms. As far as moments are concerned, the velocity and k integrations will still lead to terms which are exponentially damping following the preceding arguments. The point we wish to emphasize is that it is not adequate that velocity integrals damp rapidly (see the integrand in Eq. IV-30) as is often assumed in the literature. One must also examine the results of the K integration. We have shown that in fact the K integrations do lead to rapid decay when the integration is carried out over a range sufficiently far removed from $K = 0$.

These statements are all relevant to the validity of the usual theories of weak plasma turbulence. For instance in Kadomtsev's treatment [10] of homogeneous turbulence in a Vlasov plasma there is a driving term in the equation for the background distribution arising from the free-streaming of the initial value $f_k(v,0)$ of the form

$$\frac{\partial f}{\partial t} \doteq \frac{e}{m} \frac{\partial}{\partial v} \int dk \, \hat{E}_{-k} \, f_k(v,0) e^{i(\omega_k - kv)t} . \qquad \text{IV-35}$$

Using f to calculate the evolution of certain moments, the term on the right damps by the preceding arguments after t such that

$$k_0 \Delta t >> 1 .$$

If we were to examine Eq. IV-35 as it stands without taking velocity moments, then the most rapid growth would result from $\frac{\partial}{\partial v}$ operating on e^{-ikvt}, i.e.,

$$t \frac{e}{m} \int dk \left(-ik \hat{E}_{-k} f_k(v,0)\right) e^{i(\omega_k - kv)t} . \qquad \text{IV-36}$$

If a stationary phase analysis were applicable in Eq. IV-36 it is clear that a

$$t \times \frac{(\text{oscillation})}{t^{1/2}} \qquad \text{IV-37}$$

would ensue which diverges for large t. Using f to calculate moments, Eq. IV-35 leads to a damping contribution. For this reason we have taken the latter philosophy. In a multiple time analysis the appearance of such secularity as in Eq. IV-37 is a reflection of the fact that $t, \epsilon t, \cdots$ are not adequate time scales for a microscopic description and a more complex multiple time analysis must be done.

We can use Eq. IV-32 to construct a physical condition for the validity of the theory. The theory presented in this thesis and the usual theories for weak plasma turbulence give for the slowly changing background distribution in the region of slight instability,

$$\frac{\partial f}{\partial t} = \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial v} \int dk \, k^2 \psi_k \, \delta(\omega_k - kv) \frac{\partial f}{\partial v} . \qquad \text{IV-38}$$

From the previous discussion there are corrections to this which damp out as far as moments are concerned in times such that $k_0 \Delta t >> 1$. The basic demand we make is that

$$k_0 \Delta t_0 >> 1 , \qquad \text{IV-39}$$

where t_o is the characteristic time of variation of the moment $\int M f dv$, estimated from Eq. IV-38. In particular we consider M to have a characteristic width Δ of order the width of the instabilty region. Since Eq. IV-38 includes only the process of resonant diffusion, condition

IV-39 should give some physical estimate of the conditions for neglect of particle trapping. From Eq. IV-38

$$\frac{\partial}{\partial t}\int M f dv = -\left(\frac{e}{m}\right)^2 \iint dv\, dk \frac{\partial M}{\partial v}\frac{\partial f}{\partial v}\,\delta(\omega_k - kv)\, k^2 \psi_k .$$

Order of magnitude estimates $\left(\text{e.g.}, \frac{\partial M}{\partial v} \sim \frac{M}{\Delta}\right)$ readily give

$$\frac{\Delta}{t_o} \sim \left(\frac{e}{m}\right)^2 k_o^2 \psi_{k_o}(\Delta k) \cdot \frac{1}{k_o \Delta^2}, \qquad \text{IV-40}$$

where k_o is a typical wave number of the packet ψ_{k_o}, and Δk is of order the width of the packet. As $k_o^2 \psi_{k_o}$ is the typical energy density in the waves, $k_o^2 \psi_{k_o} \Delta k \sim E^2$ where E is the electric field amplitude. IV-40 may be written

$$\frac{1}{t_o} \sim \left(\frac{e}{m} k_o E\right)^2 \frac{1}{(k_o \Delta)^3} .$$

IV-39 becomes

$$\left(\frac{e}{m} k_o E\right)^2 \frac{1}{(k_o \Delta)^4} \ll 1 . \qquad \text{IV-41}$$

Recognizing $\left(\frac{e}{m} k_o E\right)^{1/2}$ as the approximate inverse time, ω_o, for a wave of amplitude E, wave number k_o, to trap a particle,

IV-41 can be written

$$\omega_o^4 \, \Upsilon_T^4 \ll 1 , \qquad \text{IV-42}$$

where

$$\Upsilon_T \sim \frac{1}{k_o \Delta} \sim \frac{1}{k_o \times \text{change in } \frac{\omega_k}{k} \text{ over } \Delta k}$$

$$\sim \frac{1}{k_o\, \delta\left(\frac{\omega_k}{k}\right)_{\delta k \sim \Delta k}}$$

$\sim$ transit time of particle moving through packet.

Eq. IV-42 is a statement that the transit time is short compared with the trapping time, as was previously speculated in Chapter I, and is a requirement that the field amplitude be sufficiently small and that the spectrum width be sufficiently large. Under these conditions the particle diffuses through the packet before it has time to be trapped by a particular wave of the packet.

In obtaining the above order of magnitude estimates, we have used condition IV-33 with $t \sim t_0$. As previously discussed, however, the long-wavelength phenomena undergo a slow mixing process. Thus, in order to obtain some idea of the minimum K values tolerable in the theory we ask in relation to IV-32 that

$$K_1 \Delta t_0 \gtrsim 1 .$$

Using the previous estimate of t_0 and $E^2 \sim \epsilon n k T$, this may be written

$$\frac{K_1}{K_0} \gtrsim \frac{\epsilon}{4\pi}\left(\frac{\omega_p^2/K_0^2}{\Delta^2}\right)\left(\frac{kT/m}{\Delta^2}\right).$$

K_0 is a wave-number typical of the packet, and K_1 some estimate on the lower bound. If we were to take $K_0 \sim 1/\lambda_D$, then the above estimate reads

$$(K_1 \lambda_D) \gtrsim \frac{\epsilon}{4\pi}\left(\frac{kT/m}{\Delta^2}\right)^2.$$

h) Comments on Mode Coupling

In reference to the kinetic wave equation III-65 a careful examination of the derivation in Appendix D shows that the mode coupling terms(terms associated with $\delta(\omega_k - \omega_\ell - \omega_{k-\ell})$) arise in taking the long time behavior in ℓ integrations of

$$\frac{e^{i(\omega_k - \omega_\ell - \omega_{k-\ell})t} - 1}{i(\omega_k - \omega_\ell - \omega_{k-\ell})} \quad \text{as} \quad \underset{t\to\infty}{\longrightarrow} \quad i \quad \frac{1}{\omega_k - \omega_\ell - \omega_{k-\ell} + i\delta} \; . \tag{a}$$

This is also encountered in the study of a simple equation exhibiting solely wave-wave interactions in Chapter V and is discussed in Appendix H_2 . The basic restriction found is that

$$\frac{d}{dk'}(\omega_{k'} + \omega_{k-k'}) = 0 \; , \tag{b}$$

must not be satisfied for those k' such that

$$\omega_k - \omega_{k'} = \omega_{k-k'} \; , \tag{c}$$

given k , for the theory to be valid. This is usually termed a "simple" resonance in the literature[7] in contrast to a double resonance where both (b) and (c) are satisfied. Mathematically, the $\delta(\omega_k - \omega_\ell - \omega_{k-\ell})$ arising in (a) leads to a divergent result in the ℓ integration if both (b) and (c) can be satisfied simultaneously. In Appendix H_2 we show that under these circumstances, (a) is not the proper prescription for dealing with the long-time behavior of integrals of the form

$$\int dk' \, f(k,k') \, \frac{e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} - 1}{i(\omega_k - \omega_{k'} - \omega_{k-k'})} \; ,$$

and in fact the long-time behavior diverges as $t^{1/2}$. In such situations a more general multiple time analysis than $t, \epsilon t \cdots$ must be done, since the wave-wave phenomena effectively evolve on a faster time scale. Physically, by assuming that $\frac{d}{dk'}(\omega_{k'}+\omega_{k-k'}) \neq 0$ for those k' satisfying $\omega_k - \omega_{k'} = \omega_{k-k'}$ (given k), we are excluding problems in which the wave-wave interactions are strong in the following sense: When a wave frequency ω_k decays into wave components $\omega_{k'}$ and $\omega_{k-k'}$,

$$\omega_k \rightsquigarrow \begin{array}{l} \rightsquigarrow \omega_{k'}, \left(\frac{d\omega_{k'}}{dk'}\right) \\ \rightsquigarrow \omega_{k-k'}, \left(\frac{d\omega_{k-k'}}{dk}\right) \end{array}$$

the associated group velocities are $d\omega_{k'}/dk'$ and $d\omega_{k-k'}/dk$, respectively. These group velocities are clearly equal if (b) is satisfied. Under such circumstances the associated wave disturbances move away with the same velocity and thus are capable of further interaction with one another and altering the state ($\omega_{k'}, \omega_{k-k'}$).

However, under the assumption that these group velocities are not equal, that is, $\frac{d}{dk'}(\omega_{k'}+\omega_{k-k'}) \neq 0$ for those k' such that $\omega_k - \omega_{k'} = \omega_{k-k'}$, the wave disturbances move away from one another and do not interact effectively. The argument is similar in the three-dimensional problem. In this case we limit ourselves to physical problems for which

$$\frac{d}{d\underline{k}'}\left(\omega(\underline{k}') + \omega(\underline{k}-\underline{k}')\right) \neq 0$$

for those $\underline{k}'$ satisfying

$$\omega(\underline{k}) - \omega(\underline{k}') = \omega(\underline{k}-\underline{k}').$$

i) Discrete and Continuous Representations

In the analysis presented in the previous chapters a continuous k representation has been used, consistent with the $V \to \infty$ limit. We have also envisioned a situation where a continuous band of modes have been excited as opposed to a few isolated modes. However, much of the material in the literature [7, 16] has utilized a discrete k representation in which the Four ier analysis is done in a box volume, L^3. It is ourcontention that many of these results are meaningless except in a continuum limit where $L^3 \to \infty$ and the wave numbers become closely packed. For example, in Reference 7, the kinetic equation for the spectrum ψ_k involves

$$\frac{\partial \psi_k}{\partial t} = \sum_{k'} |M_{kk'}|^2 \psi_{k'} \psi_{k-k'} \delta(\omega_k - \omega_{k'} - \omega_{k-k'}),$$

which is analogous to the last term in Eq. III-65. As it stands IV-43 is highly singular as it is a sum over delta-functions. However, in the continuum limit it may be identified with the last term in III-65.

It is important to note that changing from a discrete to a continuum representation and determining long-time behaviour are both limiting processes and that the correct order for performing such limits is somewhat ambiguous. A simple example illustrates this point. Consider the two following equations in a continuous and discrete representation.

Continuous:

$$\frac{\partial G_k}{\partial t} = \int dk' \, F(k,k') \, e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} \qquad \text{IV-44}$$

Discrete:

$$\frac{\partial G_k}{\partial t} = \sum_{k'} f(k,k')\, e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} \qquad \text{IV-45}$$

Call the set of all k' such that $\omega_k - \omega_{k'} = \omega_{k-k'}$, S.

IV-45 then yields

$$G_k = G_k(0) + t \sum_{k' \in S} f(k,k') + \sum_{k' \notin S} f(k,k') \frac{e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} - 1}{i(\omega_k - \omega_{k'} - \omega_{k-k'})} \qquad \text{IV-46}$$

IV-44 becomes

$$G_k = G_k(0) + \int dk' F(k,k') \frac{e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} - 1}{i(\omega_k - \omega_{k'} - \omega_{k-k'})} . \qquad \text{IV-47}$$

For large t (see Appendix H_2), despite the fact the integrand in IV-47 goes as t for $\omega_k - \omega_{k'} = \omega_{k-k'}$,

$$G_k \sim G_k(0) + \int dk' F(k,k') \frac{i}{(\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta)} . \qquad \text{IV-48}$$

We now examine IV-46 for large t. It can be argued that for large t the oscillating term mixes out. We are then left with

$$\sum_{k' \notin S} f(k,k') \frac{-1}{i(\omega_k - \omega_{k'} - \omega_{k-k'})} ,$$

(corresponding to the principal value, in IV-48) and a term proportional to t, i.e., $t \sum_{k' \in S} f(k,k')$.

This secular term corresponds to the $\delta(\omega_k - \omega_{k'} - \omega_{k-k'})$ in IV-48; however there is no obvious way to pass to the continuum with $t \sum_{k' \in S} f(k,k')$ as it stands. If, however, we pass to the continuum in IV-46, before taking $t \to \infty$, then upon rewriting IV-46 as

$$G_k = G_k(o) + \sum_{k'} f(k,k') \frac{e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} - 1}{i(\omega_k - \omega_{k'} - \omega_{k-k'})},$$

it is clear that IV-48 is recovered. That is to say, we obtain IV-48 if we let $L^3 \to \infty$ before $t \to \infty$. For this reason it is convenient to work in a continuum representation at the outset as in IV-44. We note that in reference to the simple example just studied, if there are no k' such that $\omega_k - \omega_{k'} = \omega_{k-k'}$, the order of $t \to \infty$, $\sum \to \int$, does not matter.

CHAPTER V

A SIMPLE EXAMPLE EXHIBITING WAVE-WAVE INTERACTIONS

a) The Basic Equation

In this chapter we wish to give a detailed analysis of a comparatively simple problem relevant to physical systems involving weak nonlinear wave-wave interactions. The structure of the equation is such as to include 3-wave processes; the simplicity of the example minimizes algebraic manipulation but retains the critical points in the analysis and results in terms similar in structure to the wave-wave terms in the kinetic equation for the waves in Chap. III. Consider the equation

$$\frac{\partial}{\partial t} E_{\underline{k}}(t) + i\omega_{\underline{k}} E_{\underline{k}}(t) = \int K(\underline{k}', \underline{k}-\underline{k}') E_{\underline{k}'}(t) E_{\underline{k}-\underline{k}'}(t)\, d\underline{k}' \qquad \text{V-1}$$

where we assume

$$\omega(-\underline{k}) = -\omega(\underline{k}) \quad ; \quad \omega(\underline{k}) \quad \text{real} \quad . \qquad \text{V-2}$$

Since we are assuming $E_{\underline{k}}$ is the Fourier transform of a real function

$$E_{-\underline{k}} = E_{\underline{k}}^{*} \quad . \qquad \text{V-3}$$

Changing variables in the integral term of Eq. V-1 from $\underline{k}'$ to $\underline{k} - \underline{k}'$ results in the same equation provided

$$K(\underline{k}', \underline{k} - \underline{k}') = K(\underline{k} - \underline{k}', \underline{k}') \quad . \qquad \text{V-4}$$

If we let $\underline{k} \longrightarrow -\underline{k}$ in Eq. V-1 and take the complex conjugate of the resulting equation, we obtain

$$\frac{\partial}{\partial t} E_{\underline{k}} + i\omega_{\underline{k}} E_{\underline{k}} = \int K^{*}(\underline{k}', -\underline{k} - \underline{k}') E^{*}_{-\underline{k} - \underline{k}'} E^{*}_{\underline{k}'}\, d\underline{k}' \quad . \qquad \text{V-5}$$

Changing variables $(\underline{k}' \longrightarrow -\underline{k}')$ in the integral term gives

$$\frac{\partial}{\partial t} E_{\underline{k}} + i\omega_{\underline{k}} E_{\underline{k}} = \int K^{*}(-\underline{k}', -\underline{k} + \underline{k}') E^{*}_{-\underline{k} + \underline{k}'} E^{*}_{-\underline{k}'}\, d\underline{k}' \quad . \qquad \text{V-6}$$

By virtue of Eq. V-3, this becomes

$$\frac{\partial E_{\underline{k}}}{\partial t} + i\omega_{\underline{k}} E_{\underline{k}} = \int K^{*}(-\underline{k}', -\underline{k}+\underline{k}')\, E_{\underline{k}-\underline{k}'}\, E_{\underline{k}'}\, d\underline{k}' \quad . \qquad \text{V-7}$$

This reproduces Eq. V-1 if

$$K(\underline{k}', \underline{k}-\underline{k}') = K^{*}(-\underline{k}', -\underline{k}+\underline{k}') \qquad \text{V-8}$$

i.e., if $\quad K(\underline{k}', \underline{k}-\underline{k}') = K^{*}(\underline{k}'-\underline{k}, -\underline{k}')$.

In summary then

$$\frac{\partial E_{\underline{k}}}{\partial t} + i\omega_{\underline{k}} E_{\underline{k}} = \int d\underline{k}'\; K(\underline{k}', \underline{k}-\underline{k}')\, E_{\underline{k}'}\, E_{\underline{k}-\underline{k}'} \quad , \qquad \text{V-9}$$

where

$$\omega(-\underline{k}) = -\omega(\underline{k}) \quad ,$$

$$E_{-\underline{k}} = E_{\underline{k}}^{*} \quad ,$$

$$K(\underline{k}', \underline{k}-\underline{k}') = K(\underline{k}-\underline{k}', \underline{k}') \quad ,$$

and

$$K^{*}(\underline{k}', \underline{k}-\underline{k}') = K(-\underline{k}', \underline{k}'-\underline{k}) \quad .$$

The evolution of a spatially homogeneous ensemble of systems each of which is described by the system of Eqs. V-9 has received much attention in the literature (see for example, Ref. 15 and Ref. 16). For example, in Ref. 16 Eq. V-9 arises in the description of a low-density, cold plasma in which the thermal motion of the particles can be neglected and the magnetohydrodynamic equations can be used to describe the motion of the plasma. (The notation "$E_{\underline{k}}$" should not in general be interpreted as denoting "electric field.")

Both Ref. 15 and Ref. 16 have in common the fact that they treat the non-linear terms as small and obtain a perturbation solution to Eq. V-9 for $\hat{E}_k$ (where $E_{\underline{k}} = \hat{E}_{\underline{k}}\, e^{-i\omega_{\underline{k}} t}$) to third order

$$\hat{E}_{\underline{k}} \simeq \lambda \hat{E}^{(1)}_{\underline{k}} + \lambda^2 \hat{E}^{(2)}_{\underline{k}} + \lambda^3 \hat{E}^{(3)}_{\underline{k}} + \ldots \qquad \text{V-10}$$

Reference 15 uses a multiple time analysis and in asking that certain ensemble quantities be nonsecular obtainsa kinetic equation for the waves. Reference 16 obtains a kinetic equation for the waves by considering the transition probability per unit time of relevant ensemble quantities. Both of these methods are briefly described in Appendix H_1 and H_3 and objections pointed out. In particular we feel that an incorrect multiple time analysis is done in Ref. 15 although the final answer is correct. We give in Appendix H_3 a derivation of the kinetic equation for the waves using techniques similar to Ref. 15, but a more rigorous multiple time approach.

The basic philosophy of both Ref. 15 and Ref. 16 is to solve Eq. V-9 to to a given order (λ^3) and <u>then</u> obtain a kinetic equation for the waves by performing appropriate averages over a statistical ensemble.

In this section we offer an alternate approach. We construct at the outset from Eq. V-9, equations advancing certain ensemble quantities and from these equations derive a kinetic equation for the waves.

We rewrite Eq. V-9 with $\hat{E}_{\underline{k}} = E_{\underline{k}}\, e^{i\omega_k t}$,

$$\frac{\partial}{\partial t} \hat{E}_{\underline{k}} = \int d\underline{k}'\; K(\underline{k}', \underline{k} - \underline{k}')\, \hat{E}_{\underline{k}'}\, \hat{E}_{\underline{k}-\underline{k}'}\, e^{i(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'})t} . \qquad \text{V-11}$$

From Eq. V-11 we can construct equations for $\hat{E}_{\underline{k}_1} \hat{E}_{\underline{k}_2}$, $\hat{E}_{\underline{k}_1} \hat{E}_{\underline{k}_2} \hat{E}_{\underline{k}_3}$, etc. Namely,

$$\frac{\partial}{\partial t} (\hat{E}_{\underline{k}_1} \hat{E}_{\underline{k}_2}) = \int d\underline{k}'\, K(\underline{k}', \underline{k}_1 - \underline{k}')\, \hat{E}_{\underline{k}'}\, \hat{E}_{\underline{k}_1-\underline{k}'}\, \hat{E}_{\underline{k}_2}\, e^{i(\omega_{\underline{k}_1} - \omega_{\underline{k}'} - \omega_{\underline{k}_1-\underline{k}'})t}$$
$$+ \int d\underline{k}'\, K(\underline{k}', \underline{k}_2 - \underline{k}')\, \hat{E}_{\underline{k}'}\, \hat{E}_{\underline{k}_2-\underline{k}'}\, \hat{E}_{\underline{k}_1}\, e^{i(\omega_{\underline{k}_2} - \omega_{\underline{k}'} - \omega_{\underline{k}_2-\underline{k}'})t} \qquad \text{V-12}$$

and

$$\frac{\partial}{\partial t}\left(E_{\underline{k}_1} E_{\underline{k}_2} E_{\underline{k}_3}\right)$$

$$= \int d\underline{k}'\, K(\underline{k}', \underline{k}_1 - \underline{k}')\, \hat{E}_{\underline{k}_1-\underline{k}'}\, \hat{E}_{\underline{k}'}\, \hat{E}_{\underline{k}_2}\, \hat{E}_{\underline{k}_3}\, e^{i(\omega_{\underline{k}_1} - \omega_{\underline{k}'} - \omega_{\underline{k}_1-\underline{k}'})t}$$

$$+ \int d\underline{k}'\, K(\underline{k}', \underline{k}_2 - \underline{k}')\, \hat{E}_{\underline{k}_2-\underline{k}'}\, \hat{E}_{\underline{k}'}\, \hat{E}_{\underline{k}_1}\, \hat{E}_{\underline{k}_3}\, e^{i(\omega_{\underline{k}_2} - \omega_{\underline{k}'} - \omega_{\underline{k}_2-\underline{k}'})t}$$

$$+ \int d\underline{k}'\, K(\underline{k}', \underline{k}_3 - \underline{k}')\, \hat{E}_{\underline{k}_3-\underline{k}'}\, \hat{E}_{\underline{k}'}, \hat{E}_{\underline{k}_1}\, \hat{E}_{\underline{k}_2}\, e^{i(\omega_{\underline{k}_3} - \omega_{\underline{k}'} - \omega_{\underline{k}_3-\underline{k}'})t} . \qquad \text{V-13}$$

$$\vdots$$

As is common on statistical theories of homogeneous turbulence it will be assumed that $E(x,t)$ is a stationary random function of position (but not of time, the time variation of average quantities being the point in question) (see Batchelor 1956).[20] We assume in averaging over the ensemble that

$$\langle \hat{E}_{\underline{k}} \rangle = 0 \quad , \qquad \text{V-14}$$

$$\langle \hat{E}_{\underline{k}_1} \hat{E}_{\underline{k}_2} \rangle = G_{\underline{k}_1}\, \delta(\underline{k}_1 + \underline{k}_2) \quad , \qquad \text{V-15}$$

$$\langle \hat{E}_{\underline{k}_1} \hat{E}_{\underline{k}_2} \hat{E}_{\underline{k}_3} \rangle = H_{\underline{k}_1, \underline{k}_2}\, \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \quad , \qquad \text{V-16}$$

$$\langle \hat{E}_{\underline{k}_1} \hat{E}_{\underline{k}_2} \hat{E}_{\underline{k}_3} \hat{E}_{\underline{k}_4} \rangle = F_{\underline{k}_1, \underline{k}_2, \underline{k}_3}\, \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4)$$

$$+ G_{\underline{k}_1} G_{\underline{k}_3}\, \delta(\underline{k}_1 + \underline{k}_2)\, \delta(\underline{k}_3 + \underline{k}_4)$$

$$+ G_{\underline{k}_1} G_{\underline{k}_2}\, \delta(\underline{k}_1 + \underline{k}_3)\, \delta(\underline{k}_2 + \underline{k}_4)$$

$$+ G_{\underline{k}_1} G_{\underline{k}_2}\, \delta(\underline{k}_1 + \underline{k}_4)\, \delta(\underline{k}_2 + \underline{k}_3) \ . \qquad \text{V-17}$$

The relevance of Eqs. V-14 and V-15 can be easily understood as follows. $\langle \hat{E}_{\underline{k}} \rangle = 0$ is a condition we impose on the system for simplicity and corresponds to the assumption that $E(x,t)$ averaged over the ensemble is zero. This is a consistent assumption for all times if true initially, provided

$$K(\underline{k}', -\underline{k}') = 0 \quad . \qquad \text{V-18}$$

This is most easily demonstrated by averaging V-11 and using Eqs. V-15 and V-18. This gives

$$\frac{\partial}{\partial t} \langle \hat{E}_{\underline{k}} \rangle = 0 \quad , \qquad \text{V-19}$$

and hence $\langle \hat{E}_{\underline{k}} \rangle = 0$ for all times if true initially.

Equations V-15 - V-17 merely reflect the spatial homogeneity of the ensemble. For example, in reference to Eq. V-15, we consider the average of the product $E(x,t)\,E(x+x_1,t)$, viz., $\langle E(x,t)\,E(x+x_1,t)\rangle$. The spatial homogeneity of the ensemble tells us that this average should depend only on the relative spatial co-ordinate x_1 , i.e., $E(x,t)$ is a stationary random function of position. In terms of Fourier transforms we ask that

$$\int d\underline{k}\, d\underline{k}'\, e^{i\underline{k}\cdot x}\, e^{i\underline{k}'\cdot(x+x_1)} \langle \hat{E}_{\underline{k}} \hat{E}_{\underline{k}'} \rangle$$

be a function only of x_1 , or that

$\langle \hat{E}_{\underline{k}} \hat{E}_{\underline{k}'} \rangle = G_k\, \delta(\underline{k}+\underline{k}')$ as given in Eq. V-15. Similarly Eq. V-16 ensures that $\langle E(x)\,E(x+x_1)\,E(x+x_2)\rangle$ depends only on x_1 and x_2 , i.e., on the relative spatial co-ordinates. In Eq. V-17 the product of 4 E's is written in terms of an irreducible four-correlation, F, and all possible products of lower correlations which in this case are GG correlations. By virtue of Eq. V-14, there is no possibility of H terms in Eq. V-17. We see that Eq. V-17 is manifestly consistent with the spatial homogeneity of the ensemble, i.e.,

$\langle E(x,t)\, E(x+x_1,t)\, E(x+x_2,t)\, E(x+x_3,t)\rangle$ depends only on x_1, x_2, and x_3. The F term allows for a general dependence on x_1, x_2, and x_3, and the GG terms a dependence on x_1, x_2, x_3, through the differences x_1, x_2-x_3; x_2, x_1-x_3; x_3, x_1-x_2. The writing of the ensemble average of the product of 4 E's in terms of an irreducible four-correlation, F, and lower correlations (in this case GG) in Eq. V-17, is similar to the usual cluster expansion employed in working with the B-B-G-K-Y hierarchy of equations. The GG terms in fact bear special significance. It is common practice in the Russian literature (e.g., References 7, 10, and 16) in averaging the product, $E^{(1)}_{k_1}\, E^{(1)}_{k_2}\, E^{(1)}_{k_3}\, E^{(1)}_{k_4}$, over an ensemble, to average with respect to the random phases of the separate oscillations by considering products in pairs, thus giving terms of the form

$$\mathcal{E}_{\underline{k}_1}\mathcal{E}_{k_3}\delta(\underline{k}_1+\underline{k}_2)\,\delta(\underline{k}_3+\underline{k}_4) + \mathcal{E}_{\underline{k}_1}\mathcal{E}_{\underline{k}_3}\delta(\underline{k}_1+\underline{k}_4)\,\delta(\underline{k}_3+\underline{k}_2)$$
$$+\ \mathcal{E}_{\underline{k}_1}\mathcal{E}_{\underline{k}_2}\,\delta(\underline{k}_1+\underline{k}_3)\,\delta(\underline{k}_2+\underline{k}_4)\ .$$

We see that this is similar to relation V-17, neglecting the effect of the irreducible 4-correlations F, and is consistent with the spatial homogeneity of the ensemble, as discussed above. A brief discussion of their averaging techniques is given in Appendix H_4.

We now return to the problem at hand and ensemble average Eqs. V-12 and V-13 utilizing V-15 - V-17. This readily gives the chain of equations for the wave correlations,

$$\delta(\underline{k}_1+\underline{k}_2)\quad \frac{\partial}{\partial t} G_{\underline{k}_1}$$
$$= \delta(\underline{k}_1+\underline{k}_2)\left\{\int d\underline{k}'\ K(\underline{k}',\underline{k}_1-\underline{k}')\, H_{\underline{k}',\,\underline{k}_1-\underline{k}'}\, e^{i(\omega_{\underline{k}_1}-\omega_{\underline{k}'}-\omega_{\underline{k}_1-\underline{k}'})t} + (1 \leftrightarrow 2)\right\} \qquad \text{V-20}$$

and

$$\delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \frac{\partial}{\partial t} H_{\underline{k}_1, \underline{k}_2}$$

$$= \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \sum_{\{1,2,3\}} \Big\{ K(-\underline{k}_3, \underline{k}_1 + \underline{k}_3) \, G_{k_2} G_{k_3} \, e^{i(\omega_{\underline{k}_1} + \omega_{\underline{k}_3} - \omega_{\underline{k}_1 + \underline{k}_3})t} + (2 \leftrightarrow 3) \Big\}$$

$$+ \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \sum_{\{1,2,3\}} \int d\underline{k}' \; K(\underline{k}', \underline{k}_1 - \underline{k}') \, F_{\underline{k}', \underline{k}_2, \underline{k}_3} \times e^{i(\omega_{\underline{k}_1} - \omega_{\underline{k}'} - \omega_{\underline{k}_1 - \underline{k}'})t} \quad , \qquad \text{V-21}$$

Consistent with the assumptions of weak nonlinearity in Eq. V-9, used in Ref. 15 and Ref. 16, we assume G_k is of order some small parameter ϵ. We observe that H is driven by GG terms and hence assume H is of order ϵ^2 to leading order, and so on.

Thus the multiple time-perturbation expansion we assume is

$$G_{\underline{k}} \simeq \epsilon G_{\underline{k}}^{(1)}(t, \epsilon t, \ldots) + \epsilon^2 G_{\underline{k}}^{(2)}(t, \epsilon t, \ldots) + \ldots \qquad \text{V-22}$$

$$H_{\underline{k}, \underline{k}'} \simeq \epsilon^2 H_{\underline{k}, \underline{k}'}^{(2)}(t, \epsilon t, \ldots) + \ldots \qquad \text{V-23}$$

$$F_{\underline{k}, \underline{k}', \underline{k}''} \simeq \epsilon^3 F_{\underline{k}, \underline{k}', \underline{k}''}^{(3)}(t, \epsilon t \ldots) + \ldots \qquad \text{V-24}$$

We will need only go to order ϵ^2 in the G equation and thus only need H to lowest order (ϵ^2). Equations V-20 and V-21 then become

$$\frac{\partial}{\partial t} G_{\underline{k}_1}^{(1)} = 0 \quad , \qquad \text{V-25}$$

$$\frac{\partial}{\partial t} G_{\underline{k}_1}^{(2)} + \frac{\partial}{\partial \epsilon t} G_{\underline{k}_1}^{(1)} = \int d\underline{k}' \; K(\underline{k}', \underline{k}_1 - \underline{k}') \, H_{\underline{k}', \underline{k}_1 - \underline{k}'}^{(2)} \, e^{i(\omega_{\underline{k}_1} - \omega_{\underline{k}'} - \omega_{\underline{k}_1 - \underline{k}'})t} + (\underline{k}_1 \rightarrow -\underline{k}_1) \quad , \qquad \text{V-26}$$

$$\frac{\partial}{\partial t} H^{(2)}_{\underline{k}_1,\underline{k}_2} = 2\, e^{i(\omega_{\underline{k}_1} + \omega_{\underline{k}_2} - \omega_{\underline{k}_1+\underline{k}_2})t}$$

$$\times \Big\{ K(\underline{k}_1 + \underline{k}_2, -\underline{k}_2)\, G^{(1)}_{\underline{k}_2}\, G^{(1)}_{-\underline{k}_1-\underline{k}_2} + K(-\underline{k}_2, -\underline{k}_1)\, G^{(1)}_{\underline{k}_1}\, G^{(1)}_{\underline{k}_2} \qquad \text{V-27}$$

$$+ K(-\underline{k}_1, \underline{k}_2 + \underline{k}_1)\, G^{(1)}_{\underline{k}_1}\, G^{(1)}_{-\underline{k}_1-\underline{k}_2} \Big\} ,$$

where in obtaining Eq. V-27 we have used the symmetry relation of Eq. V-4. We solve these equations, starting with the lowest order. Equation V-25 tells us that

$$G^{(1)}_k = G^{(1)}_k (0, \epsilon t, \ldots) .$$

In reference to Eq. V-27, we see that the GG terms appearing on the right-hand side do not vary on the t scale. The equation can therefore be integrated directly, giving

$$H^{(2)}_{\underline{k}_1,\underline{k}_2}(t, \epsilon t, \ldots) = H^{(2)}_{\underline{k}_1,\underline{k}_2}(o, \epsilon t, \ldots)$$

$$+ 2\, \frac{e^{i(\omega_{\underline{k}_1} + \omega_{\underline{k}_2} - \omega_{\underline{k}_1+\underline{k}_2})t} - 1}{i(\omega_{\underline{k}_1} + \omega_{\underline{k}_2} - \omega_{\underline{k}_1+\underline{k}_2})}$$

$$\times \Big\{ K(\underline{k}_1 + \underline{k}_2, -\underline{k}_2)\, G^{(1)}_{\underline{k}_1}(0, \epsilon t, \ldots)\, G^{(1)}_{-\underline{k}_1-\underline{k}_2} + K(-\underline{k}_2, -\underline{k}_1)\, G^{(1)}_{\underline{k}_1}\, G^{(1)}_{\underline{k}_2}$$

$$+ K(-\underline{k}_1, \underline{k}_2 + \underline{k}_1)\, G^{(1)}_{\underline{k}_1}\, G^{(1)}_{-\underline{k}_1-\underline{k}_2} \Big\} .$$

This result is then substituted into Eq. V-26. In this regard the quantity of interest is

$$H^{(2)}_{\underline{k}',\underline{k}_1-\underline{k}'} = H^{(2)}_{\underline{k}',\underline{k}_1-\underline{k}'}(0,\epsilon t,\cdots)$$

$$-2\,\frac{e^{i(\omega_{\underline{k}'}-\omega_{\underline{k}_1}-\omega_{\underline{k}'-\underline{k}_1})t}-1}{i(\omega_{\underline{k}_1}-\omega_{\underline{k}'}-\omega_{\underline{k}_1-\underline{k}'})}$$

$$\times\Big\{K(\underline{k}_1,\underline{k}'-\underline{k}_1)\,G^{(1)}_{\underline{k}_1-\underline{k}'}\,G^{(1)}_{-\underline{k}_1} + K(\underline{k}'-\underline{k}_1,-\underline{k}')\,G^{(1)}_{\underline{k}'}\,G^{(1)}_{\underline{k}_1-\underline{k}'} + K(-\underline{k}',\underline{k}_1)\,G^{(1)}_{\underline{k}'}\,G^{(1)}_{-\underline{k}_1}\Big\}. \qquad \text{V-28}$$

This gives

$$\frac{\partial}{\partial t}G^{(2)}_{\underline{k}_1} + \frac{\partial}{\partial \epsilon t}G^{(1)}_{\underline{k}_1}(0,\epsilon t,\cdots)$$

$$= 2\Bigg[\int d\underline{k}'\,\frac{e^{i(\omega_{\underline{k}_1}-\omega_{\underline{k}'}-\omega_{\underline{k}_1-\underline{k}'})t}-1}{i(\omega_{\underline{k}_1}-\omega_{\underline{k}'}-\omega_{\underline{k}_1-\underline{k}'})}\,K(\underline{k}',\underline{k}_1-\underline{k}')$$

$$\times\Big\{K(\underline{k}_1,\underline{k}'-\underline{k}_1)\,G^{(1)}_{\underline{k}_1-\underline{k}'}G^{(1)}_{-\underline{k}_1} + K(\underline{k}'-\underline{k}_1,-\underline{k}')G^{(1)}_{\underline{k}'}G^{(1)}_{\underline{k}_1-\underline{k}'} + K(-\underline{k}',\underline{k}_1)G^{(1)}_{\underline{k}'}G^{(1)}_{-\underline{k}_1}\Big\}$$

$$+ (\underline{k}_1 \to -\underline{k}_1)\Bigg]$$

$$+\Big\{\int d\underline{k}'\,K(\underline{k}',\underline{k}_1-\underline{k}')\,H^{(2)}_{\underline{k}',\underline{k}_1-\underline{k}'}(0,\epsilon t,\cdots)\,e^{i(\omega_{\underline{k}_1}-\omega_{\underline{k}'}-\omega_{\underline{k}_1-\underline{k}'})t} + (\underline{k}_1\to-\underline{k}_1)\Big\}. \qquad \text{V-29}$$

As discussed in Appendix H_2, for large t, under certain conditions, the quantity

$$\frac{e^{i(\omega_{\underline{k}}-\omega_{\underline{k}'}-\omega_{\underline{k}-\underline{k}'})t}-1}{i(\omega_{\underline{k}}-\omega_{\underline{k}'}-\omega_{\underline{k}-\underline{k}'})},$$

behaves as ~~regards~~ integrals over k' of relatively smooth functions, as

$$\frac{e^{i(\omega_{\underline{k}_1} - \omega_{\underline{k}'} - \omega_{\underline{k}_1-\underline{k}'})t} - 1}{i(\omega_{\underline{k}_1} - \omega_{\underline{k}'} - \omega_{\underline{k}_1-\underline{k}'})} \longrightarrow i \frac{1}{\omega_{k_1} - \omega_{k'} - \omega_{k_1-k'} + i\delta} .$$

$$\delta \to 0_+ \qquad \text{V-30}$$

Hence, looking at Eq. V-29, for large t and asking that $G_k^{(2)}$ not be secular in t as $t \to \infty$ (in order that our solution be uniformly valid), we have that

$$\frac{\partial G^{(1)}_{\underline{k}_1}(\epsilon t, \cdot\cdot)}{\partial \epsilon t} = 2 \int d\underline{k}' \frac{i}{\omega_{\underline{k}_1} - \omega_{\underline{k}'} - \omega_{\underline{k}_1-\underline{k}'} + i\delta} K(\underline{k}', \underline{k}_1 - \underline{k}')$$

$$\times \left\{ K(\underline{k}', \underline{k}_1 - \underline{k}') G^{(1)}_{\underline{k}_1-\underline{k}'} G^{(1)}_{-\underline{k}_1} + K(\underline{k}'-\underline{k}_1, -\underline{k}') G^{(1)}_{\underline{k}'} G^{(1)}_{\underline{k}_1-\underline{k}'} + K(-\underline{k}', \underline{k}_1) G^{(1)}_{\underline{k}'} G^{(1)}_{-\underline{k}_1} \right\}$$

$$+ (\underline{k}_1 \to -\underline{k}_1) \qquad \text{V-31}$$

barring some pathological behavior of $H^{(2)}_{\underline{k}', \underline{k}_1 - \underline{k}'}(0, \epsilon t, \ldots)$. Using the symmetry relation V-4 and V-8, Eq. V-31 may be written in a more familiar form

$$\frac{\partial G^{(1)}_{\underline{k}}}{\partial \epsilon t} = 4\pi \int d\underline{k}' \, G^{(1)}_{\underline{k}'} G^{(1)}_{\underline{k}-\underline{k}'} \, | K(\underline{k}', \underline{k}-\underline{k}') |^2 \delta(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'})$$

$$- 8 \, \mathrm{Im} \int d\underline{k}' \, G^{(1)}_{\underline{k}} G^{(1)}_{\underline{k}-\underline{k}'} \frac{K(\underline{k}', \underline{k}-\underline{k}') K(\underline{k}, \underline{k}'-\underline{k})}{(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'} + i\delta)} ;$$

$$\delta \to 0_+ \qquad \text{V-32}$$

This is identical to the kinetic equations obtained in Ref. 15 and Ref. 16. However, the technique of derivation is entirely different. We believe this method has several advantages and return to comment on this at a later point. First, however, we discuss Eq. V-32.

b) Comments on the Kinetic Equation V-32

1. If we were to interpret $G_k^{(1)}$ as the energy density in the waves (see Eq. V-15) in problems where E_k refers to the electric field amplitude, we see that V-32 describes the time evolution of this energy density due to 3-wave interactions. Namely, it describes the effect of mode coupling on the energy spectrum through the coupling of 2 waves into 1 wave or the decoupling of 1 wave into 2 for those waves for which

$$\omega_{\underline{k}} - \omega_{\underline{k}'} = \omega_{\underline{k}-\underline{k}'} \quad . \qquad \text{V-33}$$

These terms are identical in structure to the additional terms obtained in Chap. III in the case of weak turbulence in a plasma.

Relation V-33 is often difficult to satisfy for many dispersive media. (For example, in the case of plasma oscillations.) However, in situations in which it is satisfied it is the dominant wave-wave process.

2. Inherent in the above derivation is the assumption that

$$\frac{d}{dk'} (\omega_{k'} + \omega_{k-k'}) = 0 \ , \qquad \text{V-34}$$

and

$$\omega(k') + \omega(k-k') = \omega(k) \qquad \text{V-35}$$

cannot be satisfied simultaneously for some k' (given k). This has been considered in some detail in Appendix H_2 and its relation to assumption V-30 of this section pointed out. For simplicity we look at the mathematical difficulties encountered in a one-dimensional problem. In evaluating integrals of the form

$$\int dk' \quad F(k,k')\, \delta(\omega_k - \omega_{k'} - \omega_{k-k'}) \quad ,$$

we imagine changing variables from k' to $y_k = \omega_k - \omega_{k'} - \omega_{k-k'}$. Assuming this can be inverted to give $k' = k'(k, y)$, the above integral can be rewritten with appropriate limits as

$$\int dy \; \frac{F(k, k'(k, y)) \, \delta(y)}{\frac{d}{dk'}(\omega_{k'} + \omega_{k-k'})(y)} \,, \qquad V-36$$

which diverges if Eq. V-34 and V-35 can be satisfied simultaneously. In Appendix H_2 , this has been traced back to the fact that replacing

$$\frac{e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} - 1}{i(\omega_k - \omega_{k'} - \omega_{k-k'})} \quad \text{by} \quad \frac{i}{(\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta)} \,,$$

in the calculation of the long t behavior of certain k' integrals is not appropriate under these circumstances. In fact, these integrals have portions growing as $t^{1/2}$ for large t .

Consequently our derivation holds only if Eqs. V-34 and V-35 cannot be satified simultaneously. This does not seem to be a serious limitation on the theory as these conditions are difficult to satisfy in many dispersive media.

3. In problems where E_k denotes electric field amplitude, the quantity G_k can be interpreted as the electrostatic energy density. It can be shown that Eq. V-32 has the desirable property that if G_k is initially positive for the whole k spectrum it remains so. We include as in Ref. 10, the possibility of a weak instability, since this does not alter the result. Equation V-32 then becomes

$$\frac{\partial G_{\underline{k}}}{\partial \epsilon t} = 2\gamma_{\underline{k}} G_{\underline{k}}$$
$$+ 4\pi \int d\underline{k}' \, G_{\underline{k}'} G_{\underline{k}-\underline{k}'} \; |K(\underline{k}', \underline{k}-\underline{k}')|^2 \delta(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'})$$
$$- 8 \, \mathrm{Im} \int d\underline{k}' \, G_{\underline{k}} G_{\underline{k}-\underline{k}'} \; \frac{K(\underline{k}', \underline{k}-\underline{k}') \, K(\underline{k}, \underline{k}'-\underline{k})}{(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'} + i\delta)} \, . \qquad \text{V-37}$$

We assume that the $G_{\underline{k}}$ spectrum is initially positive and that the spectrum first turns negative at $\underline{k}_o$. At the instant it is going through zero

$$G_{\underline{k}_o} = 0 \, .$$

Then

$$\frac{\partial G_{\underline{k}_o}}{\partial \epsilon t} = 4\pi \int d\underline{k}' \, G_{\underline{k}'} G_{\underline{k}_o-\underline{k}'} \, |K(\underline{k}', \underline{k}_o-\underline{k}')|^2 \, \delta(\omega_{\underline{k}_o} - \omega_{\underline{k}'} - \omega_{\underline{k}_o-\underline{k}'}) \qquad \text{V-38}$$
$$> 0 \, ,$$

that is to say, it is repelled from turning negative, a useful result for the kinetics of the spectrum.

4. In many situations of interest it is convenient to introduce the number of quasi-particles $n_{\underline{k}}$, associated with the $\underline{k}$'th mode, defined by

$$n_{\underline{k}} \, \omega_{\underline{k}} = G_{\underline{k}} \, , \qquad \text{V-39}$$

In the case of a "transparent" medium[10] where $K(\underline{k}', \underline{k}-\underline{k}')$ is pure imaginary, Eq. V-32 may be rewritten in terms of $n_{\underline{k}}$ as

$$\frac{\partial n_{\underline{k}}}{\partial \epsilon t} = 4\pi \iint d\underline{k}_1 \, d\underline{k}_2 \; \frac{\delta(\underline{k} - \underline{k}_1 - \underline{k}_2) \, \delta(\omega_{\underline{k}} - \omega_{\underline{k}_1} - \omega_{\underline{k}_2})}{\omega_{\underline{k}} \, \omega_{\underline{k}_1} \, \omega_{\underline{k}_2}}$$
$$\times \left\{ |\mu(\underline{k}_1, \underline{k}_2)|^2 n_{\underline{k}_1} n_{\underline{k}_2} - 2\mu(\underline{k}_1, \underline{k}_2) \mu(\underline{k}, -\underline{k}_2) n_{\underline{k}} n_{\underline{k}_2} \right\}, \qquad \text{V-40}$$

where we have introduced the notation

$$\mu(\underline{k}_1, \underline{k}_2) = \omega_{\underline{k}_1} \omega_{\underline{k}_2} K(\underline{k}_1, \underline{k}_2) \; .$$

Under certain circumstances, namely when

$$\mu(\underline{k}_1, \underline{k}_2) = \mu(-\underline{k}, \underline{k}_2) \; , \quad \underline{k}_1 + \underline{k}_2 = \underline{k} \; , \tag{V-41}$$

Equation V-40 has the interesting integral of motion,

$$\int d\underline{k}\, n_{\underline{k}}\, \omega_{\underline{k}} \; .$$

We recognize that condition V-41 is identical in form to condition IV-19 demonstrated explicitly for the problem of weak plasma turbulence. Utilizing conditions V-2, V-4, V-8, and V-41 we may write

$$\begin{aligned} \frac{\partial}{\partial \epsilon t} \int d\underline{k}\, n_{\underline{k}}\, \omega_{\underline{k}} &= 8\pi \iiint d\underline{k}\, d\underline{k}_1\, d\underline{k}_2 \; \frac{\delta(\underline{k} - \underline{k}_1 - \underline{k}_2)\, \delta(\omega_{\underline{k}} - \omega_{\underline{k}_1} - \omega_{\underline{k}_2})}{\omega_{\underline{k}}\, \omega_{\underline{k}_1}\, \omega_{\underline{k}_2}} \\ &\qquad \times |\mu(\underline{k}_1, \underline{k}_2)|^2 n_{\underline{k}_1} n_{\underline{k}_2} (\omega_{\underline{k}_2} - \omega_{\underline{k}_1}) \\ &\equiv 0 \; . \end{aligned} \tag{V-42}$$

The right-hand side of V-42 vanishes identically as the integrand is antisymmetric under interchange of $\underline{k}_1$ and $\underline{k}_2$. A similar conservation law can be obtained for $\int d\underline{k}\;\; n_{\underline{k}}\, \underline{k}$, namely

$$\frac{\partial}{\partial \epsilon t} \int n_{\underline{k}}\, \underline{k}\, d\underline{k} = 0 \; . \tag{V-43}$$

Equations V-42 and V-43 are statements of conservation of quasi-momentum $n_{\underline{k}}\, \underline{k}$ and energy $n_{\underline{k}}\, \omega_{\underline{k}}$.

5. It is interesting to note the analogy between Eq. V-40 and the kinetic equation for phonons in a solid.[21] In fact if we write $n_{\underline{k}} \gg 1$ in this latter reference, the equation has the same form. It is of interest to note that in the case of a Rayleigh-Jeans distribution,

$$n_{\underline{k}} = T/\omega_{\underline{k}} \quad , \qquad \text{V-44}$$

where T is the effective temperature of the mode gas, the right-hand side of Eq. V-40 vanishes identically and $\partial n_k/\partial \epsilon t = 0$. Equation V-40 becomes

$$\frac{\partial n_k}{\partial \epsilon t} = 4\pi \iint d\underline{k}_1 d\underline{k}_2 \frac{\delta(\underline{k}-\underline{k}_1-\underline{k}_2)\,\delta(\omega_{\underline{k}}-\omega_{\underline{k}_1}-\omega_{\underline{k}_2})}{\omega_{\underline{k}}^2\,\omega_{\underline{k}_1}^2\,\omega_{\underline{k}_2}^2}$$

$$\times\ |\mu(\underline{k}_1,\underline{k}_2)|^2\ (\omega_{\underline{k}} - 2\omega_{\underline{k}_1})$$

$$= 4\pi \iint d\underline{k}_1 d\underline{k}_2 \frac{\delta(\underline{k}-\underline{k}_1-\underline{k}_2)\,\delta(\omega_{\underline{k}}-\omega_{\underline{k}_1}-\omega_{\underline{k}_2})}{\omega_{\underline{k}}^2\,\omega_{\underline{k}_1}^2\,\omega_{\underline{k}_2}^2}$$

$$\times\ |\mu(\underline{k}_1,\underline{k}_2)|^2\ (\omega_{\underline{k}} - \omega_{\underline{k}_1} - \omega_{\underline{k}_2})$$

$$\equiv 0 \quad ,$$

by virtue of the $\delta(\omega_{\underline{k}} - \omega_{\underline{k}_1} - \omega_{\underline{k}_2})$ in the integrand.

<u>Comments on Derivation of Kinetic Equation</u>

We wish now to comment on the advantages of the techniques of derivation of Eq. V-32 presented in this section. The one advantage is simplicity, and can most easily be seen by referring to the alternate derivation presented in Appendix H_3 . The basic philosophy in the literature has been to solve Eq. V-9 order by order,

$$\hat{E}_{\underline{k}} = \lambda \hat{E}_{\underline{k}}^{(1)} + \lambda^2 \hat{E}_{\underline{k}}^{(2)} + \ldots$$

and thereafter perform suitable averages over a spatially homogeneous ensemble and obtain a kinetic equation advancing $\langle E_{\underline{k}} E_{\underline{k}'} \rangle$. Our philosophy, however, has been to utilize Eq. V-9 at the outset to obtain equations advancing G_k , $H_{k,k'}$, etc., which characterize the spatially homogeneous ensemble and then, with the help of these equations obtain the kinetic equation advancing $G_k^{(1)}$ (Eq. V-32). That is to say, we perform the ensemble average at an earlier stage, which is a more direct way to approach the problem. This method also avoids the introduction of half powers of ϵ ($\lambda \sim \epsilon^{1/2}$). In Appendix H_3 , in the multiple time derivation of Eq. V-32 using methods similar to those in Ref. 15, the algebra is quite tedious to show that the λt time scale is superfluous in obtaining a kinetic equation, and that the quantities of interest vary on the $\lambda^2 t$ scale (i. e., the ϵ t scale). In the analysis in this section, however, the t, ϵ t,... scales occur quite naturally in that there is no apparent reason to introduce the possibility of an $\epsilon^{1/2}$ t scale. It is interesting to note that the quasi-linear analysis of the Vlasov equation in Ref. 1 indicated that the λt, $\lambda^3 t$ time scales were unnecessary.

CHAPTER VI

HIERARCHY OF EQUATIONS FOR A SPATIALLY HOMOGENEOUS ENSEMBLE OF VLASOV FLUIDS

The technique of obtaining equations for ensemble quantities at the outset, and then solving these equations in lieu of solving for the properties of an individual system and then ensemble averaging has interesting possibilities in the study of a spatially homogeneous, weakly turbulent ensemble of Vlasov fluids.

The latter approach is currently popular in the Russian literature (e. g, , Ref. 7 and Ref. 10). Namely, they solve the Vlasov equation order by order for the self-consistent field. Then they obtain a kinetic equation for the waves by averaging over a statistical ensemble assuming the phase of $E^{(1)}_{\underline{k}}$ is random. For example,

$$\langle E^{(1)}_{\underline{k}} \; E^{(1)}_{\underline{k}'} \rangle = \mathcal{E}_{\underline{k}} \; \delta\,(\underline{k} + \underline{k}') \; .$$

Their averaging procedure is a manifestation of the spatially homogeneous nature of the ensemble and is briefly discussed in Appendix H_4. We propose in this section as in the last to obtain equations advancing ensemble quantities at the outset. We imagine a collection of multispecies systems, each a continuous Vlasov fluid obeying

$$\frac{\partial f_{a_1}}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} f_{a_1} + \frac{e_{a_1}}{m_{a_1}} \underline{E}(\underline{x}_1, t) \cdot \frac{\partial f_{a_1}}{\partial \underline{v}_1} = 0 , \qquad \text{VI-1}$$

where

$$e_{a_1} \underline{E}(\underline{x}_1, t) = -\sum_{a_2} n_{a_2} \int \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} f_{a_2}(\underline{x}_2, \underline{v}_2, t)\, d\underline{x}_2\, d\underline{v}_2 \; . \qquad \text{VI-2}$$

We expect to be able to describe wave-particle and wave-wave phenomena but not the effects of particle-particle encounters associated with the discreteness of matter. In fact we will obtain a hierarchy of equations identical to equations II-9, II-11 and II-12 of Chap. II where all the terms involving the plasma parameter ϵ_p (see the relevant estimates in II-21, II-22, and II-23 of Chap. II) have been omitted. We proceed by constructing from VI-1 and VI-2, equations for the products $f_{a_1} f_{a_2}$, $f_{a_1} f_{a_2} f_{a_3}$, etc., and then performing appropriate averages of these equations for a spatially homogeneous ensemble. The averaging procedure used here is formally identical to that of Klimontovich[22] with the omission of those terms associated with the discreteness of matter. We return to illustrate this at the end of this section.

For brevity denote $d\underline{x}_2\, d\underline{v}_2$ by $d(2)$, and $f_{a_1}(x_1, v_1, t)$ by f_{a_1}.

Rewriting VI-1 and VI-2, we have

$$\frac{\partial f_{a_1}}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} f_{a_1} = \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \int d(2)\, \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} \cdot \frac{\partial}{\partial \underline{v}_1} (f_{a_1} f_{a_2}) \cdot \qquad \text{VI-3}$$

From VI-3 the equations for $f_{a_1} f_{a_2}$, $f_{a_1} f_{a_2} f_{a_3}$ are

$$\frac{\partial}{\partial t}(f_{a_1} f_{a_2}) + \left(\underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}_2}\right)(f_{a_1} f_{a_2})$$

$$= \sum_{a_3} \frac{n_{a_3}}{m_{a_1}} \int d(3) \left\{ \frac{\partial \phi_{a_1 a_3}}{\partial \underline{x}_1} \cdot \frac{\partial}{\partial \underline{v}_1} (f_{a_1} f_{a_2} f_{a_3}) + (1 \leftrightarrow 2) \right\} \qquad \text{VI-4}$$

and

$$\frac{\partial}{\partial t}(f_{a_1} f_{a_2} f_{a_3}) + \left(\underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}_2} + \underline{v}_3 \cdot \frac{\partial}{\partial \underline{x}_3}\right)(f_{a_1} f_{a_2} f_{a_3})$$

$$= \sum_{a_4} n_{a_4} \sum_{\{1,2,3\}} \int d(4)\, \frac{1}{m_{a_1}} \frac{\partial \phi_{a_1 a_4}}{\partial \underline{x}_1} \cdot \frac{\partial}{\partial \underline{v}_1} (f_{a_1} f_{a_2} f_{a_3} f_{a_4}) \,. \qquad \text{VI-5}$$

We then average VI-3 - VI-5 over an ensemble (denote $\langle\ \rangle$) introducing the cluster expansion

$$\langle f_{a_1} f_{a_2} \rangle = \langle f_{a_1} \rangle \langle f_{a_2} \rangle + G_{a_1 a_2} . \qquad \text{VI-6}$$

$$\langle f_{a_1} f_{a_2} f_{a_3} \rangle = \langle f_{a_1} \rangle \langle f_{a_2} \rangle \langle f_{a_3} \rangle + \langle f_{a_1} \rangle G_{a_2 a_3} + \langle f_{a_2} \rangle G_{a_1 a_3}$$

$$\langle f_{a_3} \rangle G_{a_1 a_2} + H_{a_1 a_2 a_3} . \qquad \text{VI-7}$$

$$\langle f_{a_1} f_{a_2} f_{a_3} f_{a_4} \rangle = \langle f_{a_1} \rangle \langle f_{a_2} \rangle \langle f_{a_3} \rangle \langle f_{a_4} \rangle + \sum \langle f_{a_1} \rangle H_{a_2 a_3 a_4}$$

$$+ \sum G_{a_1 a_2} G_{a_3 a_4} + \sum \langle f_{a_1} \rangle \langle f_{a_2} \rangle G_{a_3 a_4} + K_{a_1 a_2 a_3 a_4} . \qquad \text{VI-8}$$

The summations are over permutations of 1, 2, 3, 4 . For the case of a spatially homogeneous ensemble

$\langle f_{a_1} \rangle$ is independent of $\underline{x}_1$, VI-9

$G_{a_1 a_2}$ depends on $\underline{x}_1 - \underline{x}_2$, VI-10

$H_{a_1 a_2 a_3}$ can be taken to depend on the differences $\underline{x}_1 - \underline{x}_3$ and $\underline{x}_2 - \underline{x}_3$ since it is invariant under translation. VI-11

Averaging Eq. VI-3 gives

$$\frac{\partial}{\partial t} \langle f_{a_1} \rangle + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} \langle f_{a_1} \rangle = \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \frac{\partial}{\partial \underline{v}_1} \cdot \int \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} (\langle f_{a_1} \rangle \langle f_{a_2} \rangle + G_{a_1 a_2}) d(2). \qquad \text{VI-12}$$

By virtue of spatial homogeneity this becomes

$$\frac{\partial}{\partial t} \langle f_{a_1} \rangle = \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \int d(2) \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} \cdot \frac{\partial G_{a_1 a_2}}{\partial \underline{v}_1} , \qquad \text{VI-13}$$

where we have assumed charge neutrality

$$\sum_{a_2} n_{a_2} e_{a_2} = 0 . \qquad \text{VI-14}$$

Averaging Eq. VI-4 yields

$$\left(\frac{\partial}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}_2}\right)\left(\langle f_{a_1}\rangle\langle f_{a_2}\rangle + G_{a_1 a_2}\right)$$

$$= \sum_{a_3} n_{a_3}\left\{\frac{1}{m_{a_1}}\int d(3)\, \frac{\partial \phi_{a_1 a_3}}{\partial \underline{x}_1}\cdot\frac{\partial}{\partial \underline{v}_1}\left(\langle f_{a_1}\rangle\langle f_{a_2}\rangle\langle f_{a_3}\rangle + \langle f_{a_1}\rangle G_{a_2 a_3}\right.\right.$$

$$\left.\left. + \langle f_{a_2}\rangle G_{a_1 a_3} + \cancel{\langle f_{a_3}\rangle} G_{a_1 a_2} + H_{a_1 a_2 a_3} + (1 \leftrightarrow 2)\right)\right). \qquad \text{VI-15}$$

The $\underline{v}_i \cdot \frac{\partial}{\partial \underline{x}_i}$ operating on $\langle f_{a_i}\rangle$ vanishes by virtue of spatial homogeneity. The crossed term (/) is zero because of Eq. VI-14. Utilizing the result in Eq. VI-13 to eliminate $\langle f_{a_1}\rangle \frac{\partial}{\partial t}\langle f_{a_2}\rangle + (1 \leftrightarrow 2)$ in Eq. VI-15 gives

$$\frac{\partial}{\partial t} G_{a_1 a_2} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} G_{a_1 a_2} - \frac{1}{m_{a_1}}\frac{\partial}{\partial \underline{v}_1}\langle f_{a_1}\rangle \cdot \sum n_{a_3}\int \frac{\partial \phi_{a_1 a_3}}{\partial \underline{x}_1} G_{a_2 a_3}\, d(3)$$

$$+ \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}_2} G_{a_1 a_2} - \frac{1}{m_{a_2}}\frac{\partial}{\partial \underline{v}_2}\langle f_{a_2}\rangle \cdot \sum_{a_3} n_{a_3}\int \frac{\partial \phi_{a_2 a_3}}{\partial \underline{x}_2} G_{a_1 a_3}\, d(3)$$

$$= \sum_{a_3} n_{a_3}\left\{\frac{1}{m_{a_1}}\int \frac{\partial \phi_{a_1 a_3}}{\partial \underline{x}_1}\cdot\frac{\partial}{\partial \underline{v}_1} H_{a_1 a_2 a_3}\, d(3) + (1 \leftrightarrow 2)\right\}. \qquad \text{VI-16}$$

A similar analysis of the average of Eq. VI-5, advancing $f_{a_1} f_{a_2} f_{a_3}$, using Eqs. VI-7, -8, -13, -14, and -16 readily gives

$$\left(\frac{\partial}{\partial t} + \sum_{l=1}^{3} \underline{v}_l \cdot \frac{\partial}{\partial \underline{x}_l}\right) H_{a_1 a_2 a_3} - \sum_{a_4} n_{a_4} \sum_{\{1,2,3\}} \frac{1}{m_{a_1}}\frac{\partial}{\partial \underline{v}_1}\langle f_{a_1}\rangle \cdot \int \frac{\partial \phi_{a_1 a_4}}{\partial \underline{x}_1} H_{a_2 a_3 a_4}\, d(4)$$

$$= \sum_{a_4} n_{a_4} \sum_{\{1,2,3\}}\left\{\frac{1}{m_{a_1}}\frac{\partial G_{a_1 a_2}}{\partial \underline{v}_1}\cdot\int \frac{\partial \phi_{a_1 a_4}}{\partial \underline{x}_1} G_{a_3 a_4}\, d(4) + (2 \leftrightarrow 3)\right\}$$

$$+ \sum_{a_4} \sum_{\{1,2,3\}} \frac{n_{a_4}}{m_{a_1}} \frac{\partial}{\partial \underline{v}_1} \cdot \int \frac{\partial \phi_{a_1 a_4}}{\partial \underline{x}_1} K_{a_1 a_2 a_3 a_4}\, d(4). \qquad \text{VI-17}$$

If we truncate this hierarchy of equations by neglecting the 4-correlations K, the resulting system of equations is identical to the system II-25 - II-28 in Chap. II, provided the source term $S_{a_1 a_2}$ is neglected. That is, the system

VI-13, -16, and -17 is identical to the B-B-G-K-Y set of Eqs. II-9, II-11, and II-12 of Chap. II if the terms involving ϵ_p (see the estimates II-21, II-22, and II-23 in Chap. II) are omitted from the analysis .

We give an alternate derivation of VI-13, -16, and -17 in Appendix I. Writing $f_{a_1} = \langle f_{a_1} \rangle + \Delta f_{a_1}$ and Fourier analyzing Eq. VI-1, equations are constructed for the quantities $\langle \Delta f_{a_1}(k_1)\, \Delta f_{a_2}(k_2) \rangle$, $\langle \Delta f_{a_1}\, \Delta f_{a_2}\, \Delta f_{a_3} \rangle$, ... where the averaging is performed over a spatially homogeneous ensemble. The advantage of this approach which is completely equivalent to the derivation just presented, is that the parallel with the usual averaging techniques[20] for homogeneous turbulence employed in the previous chapter is immediately obvious. The fact that these two approaches (ensemble of Vlasov Plasmas and B-B-G-K-Y approach of Chap. II) lead to similar determinative equations, accounts for the similarity in the resulting kinetic equations appearing in the literature (see, for example, Ref. 10 and the final results of Chap. III which used the Vlasov ensemble and B-B-G-K-Y approaches, respectively) and in fact justifies the former approach.

We now outline the relations of the preceding to the Klimontovitch approach. The phase function

$$\rho_{a_1} = \frac{1}{N_{a_1}/V} \sum_{i_{a_1}=1}^{N_{a_1}} \delta(\underline{x} - \underline{x}_{i a_1})\, \delta(\underline{v} - \underline{v}_{i a_1}) \qquad \text{VI-18}$$

satisfies the equation

$$\frac{\partial}{\partial t} \rho_{a_1} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} \rho_{a_1}$$

$$= \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \int \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}} (|\underline{x} - \underline{x}'|) \cdot \frac{\partial \rho_{a_1}(\underline{x}, \underline{v}, t)}{\partial \underline{v}} \qquad \text{VI-19}$$

$$\times \left\{ \rho_{a_2}(\underline{x}', \underline{v}', t) - \frac{1}{n_{a_2}} \delta(\underline{x} - \underline{x}')\, \delta(\underline{v} - \underline{v}') \right\} d\underline{x}' d\underline{v}',$$

by virtue of the classical equations of motion

$$\frac{d}{dt}\underline{x}_i(t) = \underline{v}_i(t) \quad ; \quad m_i \frac{d\underline{v}_i}{dt} = -\sum_{\substack{j \\ j\neq i}} \frac{\partial \phi_{ij}}{\partial \underline{x}_i} \quad . \qquad \text{VI-20}$$

Except for the δ -function term appearing in the integral of VI-19, which is related to the discreteness of matter, Eq. VI-19 is identical in form to the Vlasov Eq. VI-1; and with the omission of this discreteness term, equations for $\rho_{a_1}\rho_{a_2}$, $\rho_{a_1}\rho_{a_2}\rho_{a_3}\cdots$ can be constructed similar in form to VI-4 and VI-5. Using the explicit form of ρ in Eq. VI-19, the definition of the s-particle reduced distribution

$$f_{a_1 a_2 \cdots a_s}(1, 2, \ldots, s) = V^s \int f_{N_{a_1} N_{a_2}} \cdots \; d(s+1) \ldots \qquad \text{VI-21}$$

where $f_{N_{a_1} N_{a_2}}$ is the Liouville distribution, and the normalization condition

$$1 = \int f_{N_{a_1} N_{a_2}} \cdots \; d(1) \ldots \quad , \qquad \text{VI-22}$$

it is straightforward to show[22] in the N-V limit by averaging over $f_{N_{a_1} N_{a_2}} \cdots$,

$$f_{a_1}(1) = \overline{\rho_{a_1}(1)} \qquad \text{VI-23}$$

$$f_{a_1 a_2}(1,2) = \overline{\rho_{a_1}(1)\left(\rho_{a_2}(2) - \frac{\delta_{a_1 a_2}}{n_{a_2}}\,\delta(\underline{x}_1-\underline{x}_2)\,\delta(\underline{v}_1-\underline{v}_2)\right)} \qquad \text{VI-24}$$

$$f_{a_1 a_2 a_3}(1,2,3) = \overline{\rho_{a_1}(1)\left(\rho_{a_2}(2) - \frac{\delta_{a_1 a_2}}{n_{a_2}}\,\delta(\underline{x}_1-\underline{x}_2)\,\delta(\underline{v}_1-\underline{v}_2)\right)}$$

$$\overline{\times\left(\rho_{a_3}(3) - \frac{\delta_{a_1 a_3}}{n_{a_3}}\,\delta(\underline{x}_1-\underline{x}_3)\,\delta(\underline{v}_1-\underline{v}_3) - \frac{\delta_{a_2 a_3}}{n_{a_3}}\,\delta(\underline{x}_2-\underline{x}_3)\,\delta(\underline{v}_2-\underline{v}_3)\right)} \qquad \text{VI-25}$$

$$f_{a_1a_2a_3a_4}(1,2,3,4) = \overline{\rho_{a_1}(1)\left(\rho_{a_2}(2) - \frac{\delta_{a_1a_2}}{n_{a_2}}\delta(\underline{x}_1-\underline{x}_2)\delta(\underline{v}_1-\underline{v}_2)\right)}$$

$$\overline{\times\left(\rho_{a_3}(3) - \frac{\delta_{a_1a_3}}{n_{a_3}}\delta(\underline{x}_1-\underline{x}_3)\delta(\underline{v}_1-\underline{v}_3) - \frac{\delta_{a_2a_3}}{n_{a_3}}\delta(\underline{x}_2-\underline{x}_3)\delta(\underline{v}_2-\underline{v}_3)\right)}$$

$$\overline{\times\left(\rho_{a_4}(4) - \frac{\delta_{a_1a_4}}{n_{a_4}}\delta(\underline{x}_1-\underline{x}_4)\delta(\underline{v}_1-\underline{v}_4) - \frac{\delta_{a_2a_4}}{n_{a_4}}\delta(\underline{x}_2-\underline{x}_4)\delta(\underline{v}_2-\underline{v}_4) - \frac{\delta_{a_3a_4}}{n_{a_4}}\delta(\underline{x}_3-\underline{x}_4)\delta(\underline{v}_3-\underline{v}_4)\right)}. \qquad \text{VI-26}$$

If we omit the δ-function terms in VI-24 and VI-26 then

$$\overline{\rho_{a_1}(1)\,\rho_{a_2}(2)} \simeq f_{a_1a_2}(1,2)\ , \qquad \text{VI-27}$$

$$\overline{\rho_{a_1}(1)\,\rho_{a_2}(2)\,\rho_{a_3}(3)} \simeq f_{a_1a_2a_3}(1,2,3)\ , \qquad \text{VI-28}$$

$$\overline{\rho_{a_1}(1)\,\rho_{a_2}(2)\,\rho_{a_3}(3)\,\rho_{a_4}(4)} \simeq f_{a_1a_2a_3a_4}(1,2,3,4)\ . \qquad \text{VI-29}$$

With the definitions of f_2, f_3, and f_4 in terms of irreducible correlations g, h, and k (Eqs. II-6, II-7, and II-8 of Chap. II), we see that VI-27, -28, and -29 are of a form identical to VI-6, -7, and -8 of this section and hence Eqs. VI-13, -16, and -17 are obtained for $f_{a_1}(1)$, $g_{a_1,a_2}(1,2)$, and $h_{a_1a_2a_3}(1,2,3)$.

If the δ-function terms had been retained in VI-19, -24, -25, and -26, the complete equations for f, g, and h (Eqs. II-9, II-11, and II-12 of Chap. II) would have been obtained. Thus, in the context of the estimates made in Chap. II (II-22 and II-23 of Chap. II), the omission of these "discreteness" terms has been formally the same thing as taking $\epsilon_p \rightarrow 0$. Alternatively this may be viewed as removing the effects of single-particle encounters by subdividing the charged particles into smaller and smaller units[23] such that

$$\begin{array}{ll} e' \longrightarrow 0 & \\ m \longrightarrow 0 & n\,e \longrightarrow \text{const} \\ 1/n \longrightarrow 0 & n\,m \longrightarrow \text{const} \end{array}$$

in Eqs. II-9, II-11, and II-12 of Chap. II.

Although the analysis in this chapter has been done for a spatially homogeneous ensemble, it is obvious that the conclusions are the same if the ensemble is spatially inhomogeneous. In lieu of VI-13, VI-16, and VI-17, the spatially inhomogeneous versions of II-9, II-11, and II-12 are obtained with the omission of the same discreteness terms associated with single-particle encounters.

APPENDIX A

Coulomb Interaction Energy Density

The quantity $\sum_{\alpha,\beta} (4\pi n_\alpha e_\alpha)(4\pi n_\beta e_\beta) \iint d\underline{v}_1 d\underline{v}_2 \; g_{\alpha\beta}(\underline{k},\underline{v}_1,\underline{v}_2,t)$ 1

bears special significance in reference to the electrostatic energy in a plasma. We consider in the N-V limit, the ensemble average of energy per unit volume due to coulomb interactions in a multicomponent spatially homogeneous plasma. We assume charge neutrality, namely

$$\sum_\alpha n_\alpha e_\alpha = 0 \,. \tag{2}$$

We have

$$\lim_{V\to\infty} \frac{1}{V} \left\langle \frac{1}{2} \sum_{\alpha,\beta} \sum_{\substack{i,j \\ i\neq j}} \frac{e_\alpha e_\beta}{|\underline{x}_i - \underline{x}_j|} \right\rangle_{\text{ENS. AV.}}$$

$$= \lim_{V\to\infty} \frac{1}{2} \sum_{a_1,a_2} \sum_{\substack{i,j \\ i\neq j}} \frac{1}{V} \int \frac{e_{a_1} e_{a_2}}{|\underline{x}_i - \underline{x}_j|} f_{N_{a_1} N_{a_2}\cdots} \; d\underline{x}_1^{a_1} \cdots d\underline{x}_j^{a_2} \cdots$$

$$= \lim_{V\to\infty} \frac{1}{2} \sum_{a_1,a_2} n_{a_1} e_{a_1} n_{a_2} e_{a_2} \frac{1}{V} \int \frac{d\underline{x}_1 \, d\underline{x}_2 \, d\underline{v}_1 d\underline{v}_2}{|\underline{x}_1 - \underline{x}_2|} f_{a_1 a_2}(1,2) \,. \tag{3}$$

We write $f_{a_1 a_2}(1,2,) = f_{a_1}(1) \; f_{a_2}(2) + g_{a_1 a_2}(1,2)$. 4

Because of spatial homogeneity

$$f_{a_1}(1) = f_{a_1}(\underline{v}_1) \,, \tag{5}$$

and

$$g_{a_1,a_2}(1,2) = g_{a_1 a_2}(\underline{x}_1 - \underline{x}_2, \underline{v}_1, \underline{v}_2) . \tag{6}$$

Also

$$\int d\underline{v}_1 \; f_{a_1}(1) = 1 \,. \tag{7}$$

Enforcing the charge neutrality condition (2), (3) may be written using (4)-(7), as

$$\lim_{V\to\infty} \frac{1}{2} \sum_{a_1,a_2} n_{a_1} n_{a_2} e_{a_1} e_{a_2} \frac{1}{V} \int \frac{d\underline{x}_1 d\underline{v}_1 \, d\underline{x}_2 d\underline{v}_2}{|\underline{x}_1 - \underline{x}_2|} g_{a_1 a_2} \tag{8}$$

$$= \frac{1}{2} \sum_{a_1,a_2} n_{a_1} e_{a_1} n_{a_2} e_{a_2} \int \frac{d\underline{x}}{|\underline{x}|} d\underline{v}_1 d\underline{v}_2 \; g_{a_1 a_2}(\underline{x},\underline{v}_1,\underline{v}_2,t) .$$

Thus, the energy density of coulomb interaction may be written

$$\mathcal{E} = \frac{1}{(8\pi)} \int \frac{d\underline{k}}{(2\pi)^3} \frac{1}{k^2} \Big(\sum_{a_1,a_2} (4\pi n_{a_1} e_{a_1})(4\pi n_{a_2} e_{a_2})$$

$$\times \iint d\underline{v}_1 d\underline{v}_2 \; g_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2,t) \Big) .$$

Hence

$$\frac{1}{(8\pi)k^2}\sum_{a_1,a_2}(4\pi n_{a_1}e_{a_1})(4\pi n_{a_2}e_{a_2})\iint d\underline{v}_1\, d\underline{v}_2\; g_{a_1a_2}(\underline{k},\underline{v}_1,\underline{v}_2,t)$$

can be interpreted as the energy density of coulomb interaction in

$\underline{k}$- space.

APPENDIX B

Formal Solution for $g^{(1)}_{a,a_2}(s)$

Consider Eq. III-35

$$\left(\frac{\partial}{\partial t} + L^{(0)}_{a_1}(\underline{k},\underline{v}_1) + L^{(0)}_{a_2}(-\underline{k},\underline{v}_2) \right) g^{(1)}_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2,t,\epsilon t,\cdots) = 0, \quad 1$$

where

$$L^{(0)}_{a_1}(\underline{k},\underline{v}_1) \equiv \iota \underline{k}\cdot\underline{v}_1 - \frac{4\pi e_{a_1}}{m_{a_1}} \frac{\iota \underline{k}}{k^2}\cdot\frac{\partial f^{(0)}_{a_1}}{\partial \underline{v}_1} \sum_{a_1} n_{a_1} e_{a_1} \int d\underline{v}_1 \,, \quad 2$$

and $f^{o}_{a_1}$ does not vary on the t scale by virtue of III-32. We denote the Laplace transform of $g(t)$ by $g(s)$, i.e.,

$$g(s) = \int_0^\infty g(t)\, e^{-st} dt \,, \quad 3$$

where, provided $g(t)$ does not grow faster than e^{at}, $g(s)$ is analytic in the region for Re $s > a$. Similarly the inverse of $g(s)$ is defined as

$$g(t) = \int_C ds/(2\pi i)\; g(s) e^{st} \,, \quad 4$$

where the contour C is parallel to the Re $s=0$ axis, but to the right of all the singularities of $g(s)$. For simplicity we denote

$$L_{a_1} \equiv L^{(0)}_{a_1}(\underline{k},\underline{v}_1) \quad 5$$

We note that L_{a_1} is just the usual Landau operator encountered in Landau's solution of the linearized Vlasov equation, viz.

$$\left(\frac{\partial}{\partial t} + L^{(0)}_{a_1}(\underline{k},\underline{v}_1) \right) f_{a_1}(\underline{k},\underline{v}_1,t) = 0 \,. \quad 6$$

In reference to 1, we may write a purely formal solution in terms of the operators L_{a_1} and L_{a_2}, namely

$$g^{(1)}_{a_1 a_2}(t) = e^{-L_{a_1} t} e^{-L_{a_2} t} g^{(1)}_{a_1 a_2}(0) \,. \quad 7$$

Now the operator $e^{-L_{a_1}t}$ has the Laplace transform $\frac{1}{s + L_{a_1}}$ and thus we can represent

$$e^{-L_{a_1}t} \equiv \frac{1}{(2\pi i)} \int_{C_1} \frac{ds_1 \, e^{s_1 t}}{s_1 + L_{a_1}} . \qquad 8$$

Hence

$$g^{(1)}_{a_1 a_2}(t) = \frac{1}{(2\pi i)^2} \int_{C_1} \frac{ds_1 e^{s_1 t}}{s_1 + L_{a_1}} \int_{C_2} \frac{ds_2 \, e^{s_2 t}}{s_2 + L_{a_2}} \quad g^{(1)}_{a_1 a_2}(0) . \qquad 9$$

The order of the operators $\frac{1}{s_1 + L_{a_1}}$, $\frac{1}{s_2 + L_{a_2}}$, does not matter since L_{a_1} and L_{a_2} commute. C_1 and C_2 are the usual Laplace contours taken to the right of the singularities in the corresponding integrands. We note from 9 that we may write

$$g^{(1)}_{a_1 a_2}(s) = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{ds_1 \, ds_2}{s - s_1 - s_2} \; \frac{1}{s_1 + L_{a_1}} \; \frac{1}{s_2 + L_{a_2}} \quad g^{(1)}_{a_1 a_2}(0)$$

where $\mathrm{Re}\, s > \mathrm{Re}(s_1 + s_2)$ for s_1 on C_1 and s_2 on C_2. The problem then reduces to a knowledge of the action of the operators $\frac{1}{s + L_{a_i}}$. This however is known from the solution of the Landau problem, namely

$$\frac{1}{s_1 + L^{(0)}_{a_1}(\underline{k}, \underline{v}_1)} \quad g^{(1)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, 0, \epsilon t \cdots)$$

$$\equiv \frac{1}{s_1 + i\underline{k}\cdot\underline{v}_1} \left(1 + \frac{4\pi e_{a_1}/m_{a_1}}{\epsilon(\underline{k}, s_1)} \; \frac{i\underline{k}}{k^2} \cdot \frac{\partial f^{(0)}_{a_1}}{\partial \underline{v}_1} \sum_{a_1 = 1}^{\alpha} n_{a_1} e_{a_1} \int \frac{d\underline{v}_1}{s_1 + i\underline{k}\cdot\underline{v}_1} \right)$$

$$\times \; g^{(1)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, 0, \epsilon t \cdots) \quad .$$

APPENDIX C_1

Long-Time Behaviour of $g^{(1)}_{a_1 a_2}$

To obtain the expression for $g^{(1)}_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2,t)$ as a distribution where

$$g^{(1)}_{a_1 a_2}(t) = \frac{1}{(2\pi i)^2}\int_{C_1}\frac{e^{s_1 t}ds_1}{s_1 + L^{(0)}_{a_1}(\underline{k},\underline{v}_1)}\int_{C_2}\frac{e^{s_2 t}ds_2}{s_2 + L^{(0)}_{a_2}(-\underline{k},\underline{v}_2)}\, g_{a_1 a_2}(\underline{k},\underline{v}_1,\underline{v}_2,0,et\cdot\cdot) \qquad 1$$

we first consider the inversion of

$$\frac{1}{(2\pi i)}\int_{C_1}\frac{e^{s_1 t}ds_1}{s_1 + L^{(0)}_{a_1}(\underline{k},\underline{v}_1)}\, f_{\underline{k}}(\underline{v}_1)\,, \qquad 2$$

where $f_{\underline{k}}(\underline{v}_1)$ is analytic in a strip containing $\underline{v}_1$ real. Multiplying this by $\oplus(\underline{v}_1)$ and interpreting over $\underline{v}_1$, and changing the order of integration gives for (2)

$$\frac{1}{(2\pi i)}\int_{C_1} ds_1 e^{s_1 t}\int d\underline{v}\,\frac{\oplus(\underline{v})\, f_{\underline{k}}(\underline{v})}{s_1 + i\underline{k}\cdot\underline{v}_1}$$

$$+\frac{1}{(2\pi i)}\int_{C_1}\frac{ds_1 e^{s_1 t}}{\epsilon(\underline{k},s_1)}\int d\underline{v}_1\,\frac{\frac{e_{a_1}}{m_{a_1}}\frac{i\underline{k}}{k^2}\cdot\frac{\partial f^{0}_{a_1}}{\partial \underline{v}_1}\oplus(\underline{v}_1)}{s_1 + i\underline{k}\cdot\underline{v}_1}\sum_{a_1}4\pi n_{a_1}e_{a_1}\int\frac{d\underline{v}\, f_{k}(\underline{v},0)}{s + i\underline{k}\cdot\underline{v}} \qquad 3$$

By virtue of the analytic properties assumed, the $-i\underline{k}\cdot\underline{v}$ poles lead to damping terms. Picking up the contribution of the zero of $\epsilon(\underline{k},s_1)$ at $s_1 = -i\omega_{\underline{k}}$ gives for 3, for large t

$$e^{-i\omega_{\underline{k}} t}\int\frac{e_{a_1}}{m_{a_1}}\,\frac{\frac{i\underline{k}}{k^2}\cdot\frac{\partial f^{(0)}_{a_1}}{\partial \underline{v}_1}\oplus(\underline{v}_1)\,d\underline{v}_1}{-i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}_1 + \Delta}\sum_{a_1}\frac{4\pi n_{a_1}e_{a_1}}{\left.\frac{\partial \mathcal{E}_{\underline{k}}}{\partial s}\right|_{s=-i\omega_{k}}}\int\frac{d\underline{v}\, f_{\underline{k}}(\underline{v},0)}{-i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v} + \Delta} \qquad 4$$

The Laplace contour is pulled over into the left-hand plane (with Re $C_1 < 0$) with the proper analytic continuation of the functions involved and the oscillating pole $\omega_{\underline{k}}$ picked up. It is assumed that contribution to the integral along C_1, $\rightarrow 0$ for large t. We see that

$$\frac{1}{(2\pi i)}\int_{C_1}\frac{ds_1\, e^{s_1 t}}{s_1 + L^{(0)}_{a_1}(\underline{k},\underline{v}_1)} \equiv \frac{e_{a_1}}{m_{a_1}}\,\frac{\frac{i\underline{k}}{k^2}\cdot\partial f^{(0)}_{a_1}/\partial\underline{v}_1}{-i\omega_{\underline{k}} + i\underline{k}.\underline{v}_1 + \Delta}\, e^{-i\omega_{\underline{k}} t}\sum_{a_1}\frac{4\pi n_{a_1}e_{a_1}}{\left.\frac{\partial\varepsilon_{\underline{k}}}{\partial s}\right|_{s=-i\omega_k}}\int\frac{d\underline{v}_1}{-i\omega_k + i\underline{k}.\underline{v}_1 + \Delta} \tag{5}$$

and thus that

$$g^{(1)}_{a_1a_2} = \frac{e_{a_1}}{m_{a_1}}\,\frac{i\underline{k}\cdot\partial f^{0}_{a_1}/\partial v_1}{-i\omega_{\underline{k}} + i\underline{k}.\underline{v}_1 + \Delta}\,\frac{e_{a_2}}{m_{a_2}}\,\frac{-i\underline{k}\cdot\partial f^{0}_{a_2}/\partial v_2}{i\omega_{\underline{k}} - i\underline{k}.\underline{v}_2 + \Delta}\,\Psi_{\underline{k}}, \tag{6}$$

by virtue of 1 and 5

where

$$\Psi_{\underline{k}} = \frac{1}{k^4}\sum_{a_1,a_2}\frac{(4\pi n_{a_1}e_{a_1})(4\pi n_{a_2}e_{a_2})}{\left|\frac{\partial\varepsilon_k}{\partial s}\right|^2_{s=-i\omega_k}}\iint\frac{d\underline{v}_1\, d\underline{v}_2\, g_{a_1a_2}(\underline{k},\underline{v}_1,\underline{v}_2,0,\epsilon t\ldots)}{(-i\omega_k + i\underline{k}.\underline{v}_1+\Delta)(i\omega_k - i\underline{k}.\underline{v}_2+\Delta)} \cdot \tag{7}$$

$\Psi_{\underline{k}}$ is $\frac{8\pi}{k^2}$ times the lowest order energy density (associated with $g^{(1)}_{a_1a_2}$) of coulomb interaction, for if we consider expression 6, then

$$\frac{1}{(8\pi)k^2}\sum_{a_1,a_2}\iint d\underline{v}_1 d\underline{v}_2\; g^{(1)}_{a_1a_2}\,(4\pi n_{a_1}e_{a_1})(4\pi n_{a_2}e_{a_2}) \tag{8}$$

$$= \frac{k^2}{(8\pi)}\,\big(1 - \epsilon(\underline{k}, -i\omega_{\underline{k}}+\Delta)\big)\big(1 - \epsilon(-\underline{k}, i\omega_{\underline{k}}+\Delta)\big)\,\Psi_{\underline{k}}$$

$$= \frac{k^2}{(8\pi)}\,\Psi_{\underline{k}}$$

This can also be seen directly from the original expression for $g^{(1)}_{a_1 a_2}$, given in 1.

We first note the operator identity

$$\sum_{a_1} (4\pi n_{a_1} e_{a_1}) \int \frac{d\underline{v}_1}{s_1 + L^{(0)}_{a_1}(\underline{k}, \underline{v}_1)} \equiv \frac{1}{\epsilon(\underline{k}, s_1)} \sum_{a_1} (4\pi n_{a_1} e_a) \int \frac{d\underline{v}_1}{s_1 + i\underline{k}\cdot\underline{v}_1} . \quad 9$$

Operating directly on 1 with $\frac{1}{k^4} \sum_{a_1, a_2} (4\pi n_{a_1} e_a)(4\pi n_{a_2} e_{a_2}) \iint d\underline{v}_1 \, d\underline{v}_2$ then gives

$$\frac{1}{k^4} \sum_{a_1, a_2} (4\pi n_{a_1} e_a)(4\pi n_{a_2} e_{a_2}) \frac{1}{(2\pi i)^2} \int_{C_1}\int_{C_2} \frac{ds_1 \, ds_2 \, e^{s_1 t} e^{s_2 t}}{\epsilon(\underline{k}, s_1)\, \epsilon(-\underline{k}, s_2)} \iint \frac{d\underline{v}_1 \, d\underline{v}_2 \, g^{(1)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, 0, \epsilon t)}{(s_1 + i\underline{k}\cdot\underline{v}_1)\ (s_2 - i\underline{k}\cdot\underline{v}_2)}$$

Picking up the contributions from the poles at $s_1 = -i\omega_{\underline{k}}$ and $s_2 = i\omega_{\underline{k}}$, by pulling the Laplace contour over as before gives

$$\frac{1}{k^4} \frac{1}{\left|\frac{\partial \mathcal{E}_k}{\partial s}\right|^2_{s=-i\omega_k}} \sum_{a_1, a_2} (4\pi n_{a_1} e_{a_1})(4\pi n_{a_2} e_{a_2}) \iint \frac{d\underline{v}_1 \, d\underline{v}_2 \ g^{(1)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, 0, \epsilon t \cdots)}{(-i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}_1 + \Delta)(i\omega_{\underline{k}} - i\underline{k}\cdot\underline{v}_2 + \Delta)}$$

$$\equiv \Psi_{\underline{k}} .$$

APPENDIX C_2

Solution of Eq. III-55 in terms of Laplace Variables

To understand the action of $e^{-L^{(0)}_{a_2}(\underline{k}_2,\underline{v}_2)t}$ on $g_{a_2}(\underline{k}_2,\underline{v}_2)$ we express it in terms of its Laplace transform; namely,

$$\frac{1}{(2\pi i)}\int_{C_2}\frac{ds_2\, e^{s_2 t}}{s_2+L^{(0)}_{a_2}(\underline{k}_2,\underline{v}_2)}\, g_{a_2}(\underline{k}_2,\underline{v}_2), \qquad 1$$

where

$$\frac{1}{s_2+L^{(0)}_{a_2}(\underline{k}_2,\underline{v}_2)} = \frac{1}{s_2+i\underline{k}_2\cdot\underline{v}_2}\left[1 + \frac{e_{a_2}}{m_{a_2}k_2^2}\,\frac{i\underline{k}_2\cdot\partial f^{(0)}_{a_2}/\partial\underline{v}_2}{\epsilon(\underline{k}_2,s_2)}\sum_{a_2}4\pi n_{a_2}e_{a_2}\int\frac{d\underline{v}_2}{s_2+i\underline{k}_2\cdot\underline{v}_2}\right].$$

We first observe that

$$\sum_{a_2}(4\pi n_{a_2}e_{a_2})\int d\underline{v}_2\;\frac{g_{a_2}(\underline{k}_2,\underline{v}_2)}{s_2+i\underline{k}_2\cdot\underline{v}_2} \qquad 2$$

$$=\sum_{a_2}\frac{4\pi n_{a_2}e_{a_2}^2}{m_{a_2}}\int d\underline{v}_2\;\frac{i\underline{k}_2\cdot\partial f^{(0)}_{a_2}/\partial\underline{v}_2}{(s_2+i\underline{k}_2\cdot\underline{v}_2)(-i\omega_{k_2}+i\underline{k}_2\cdot\underline{v}_2+\Delta)}$$

Writing $\dfrac{1}{s_2+i\underline{k}_2\cdot\underline{v}_2}\times\dfrac{1}{-i\omega_{k_2}+i\underline{k}_2\cdot\underline{v}_2+\Delta}$ in terms of partial fractions, using the expression for $\epsilon(\underline{k}_2,s_2)$ and the fact that $\epsilon(\underline{k}_2,-i\omega_{k_2}+\Delta)=0$, 2 readily becomes

$$\frac{\epsilon(\underline{k}_2,s_2)}{s_2+i\omega_{k_2}-\Delta} \qquad 3$$

We thus have that

$$\frac{1}{s_2+L^{(0)}_{a_2}(\underline{k}_2,\underline{v}_2)}\,g_{a_2}(\underline{k}_2,\underline{v}_2) = \frac{g_{a_2}(\underline{k}_2,\underline{v}_2)}{s_2+i\underline{k}_2\cdot\underline{v}_2} + \frac{e_{a_2}}{m_{a_2}}\,\frac{i\underline{k}_2\cdot\partial f^{(0)}_{a_2}/\partial\underline{v}_2}{(s_2+i\underline{k}_2\cdot\underline{v}_2)(s_2-i\omega_{k_2}+\Delta)}$$

which simply reduces to $\dfrac{g_{a_2}(\underline{k}_2,\underline{v}_2)}{s_2+i\omega_{k_2}}$, and hence

$$e^{-L^{(0)}_{a_2}(\underline{k}_2,\underline{v}_2)t}\,g_{a_2}(\underline{k}_2,\underline{v}_2) = e^{-i\omega_{k_2}t}\,g_{a_2}(\underline{k}_2,\underline{v}_2). \qquad 4$$

We thus see that $g_{a_2}(\underline{k}_2, \underline{v}_2)$ is the eigenfunction of the operator $L^{(o)}_{a_2}(\underline{k}_2, \underline{v}_2)$, having as eigenvalue $i\omega_{\underline{k}_2}$. The operator

$$e^{-(L_{a_1}+L_{a_2}+L_{a_3})t}\int_0^t dt'\, e^{(L_{a_1}+L_{a_2}+L_{a_3})t'} g_{a_2}(\underline{k}_2,\underline{v}_2)\, g_{a_3}(\underline{k}_3,\underline{v}_3),$$

appearing in Eq. III-55, thus becomes

$$\int_0^t dt'\, e^{(L_{a_1}+i\omega_{\underline{k}_2}+i\omega_{\underline{k}_3})(t'-t)} g_{a_2}(\underline{k}_2,\underline{v}_2)\, g_{a_3}(\underline{k}_3,\underline{v}_3) .$$

The Laplace transform of this is simply

$$\frac{g_{a_2}(\underline{k}_2,\underline{v}_2)\, g_{a_3}(\underline{k}_3,\underline{v}_3)}{s\left(s+i\omega_{\underline{k}_2}+i\omega_{\underline{k}_3}+L^{(o)}_{a_1}(\underline{k}_1,\underline{v}_1)\right)} .$$

APPENDIX C_3

Effect of the Initial Value $h^{(2)}_{a_1 a_2 a_3}(\underline{\ell}, -\underline{k}, \underline{v}_1, \underline{v}_2, \underline{v}_3, 0, \epsilon t \cdots)$

The initial value portion of the solution for $h^{(2)}$ (see also Appendix C_1) may be written

$$h^{(2)}_{a_1 a_2 a_3}(\underline{k}_1, \underline{k}_2, \underline{v}_1, \underline{v}_2, \underline{v}_3, t, \epsilon t \cdots)\,\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3) \qquad 1$$
$$= \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3)\, g_{a_1}(\underline{k}_1, \underline{v}_1)\, g_{a_2}(\underline{k}_2, \underline{v}_2)\, g_{a_3}(\underline{k}_3, \underline{v}_3)\, \psi_{k_1 k_2 k_3}\, e^{-i(\omega_{k_1}+\omega_{k_2}+\omega_{k_3})t}$$

where

$$\psi_{\underline{k}_1 \underline{k}_2 \underline{k}_3} = \sum_{a_1, a_2 a_3} \frac{(4\pi n_{a_1} e_{a_1})(4\pi n_{a_2} e_{a_2})(4\pi n_{a_3} e_{a_3})}{\left.\frac{\partial \mathcal{E}_{k_1}}{\partial s}\right|_{s=-i\omega_{k_1}} \left.\frac{\partial \mathcal{E}_{k_2}}{\partial s}\right|_{s=-i\omega_{k_2}} \left.\frac{\partial \mathcal{E}_{k_3}}{\partial s}\right|_{s=-i\omega_{k_3}}}$$
$$\times \iiint \frac{d\underline{v}_1\, d\underline{v}_2\, d\underline{v}_3\; h^{(2)}_{a_1 a_2 a_3}(k_1, k_2, v_1, v_2, v_3, 0, \epsilon t \cdots)}{(-i\omega_{k_1} + ik_1 \cdot v_1 + \Delta)(-i\omega_{\underline{k}_2} + i\underline{k}_2 \cdot \underline{v}_2 + \Delta)(-i\omega_{k_3} + i\underline{k}_3 \cdot \underline{v}_3 + \Delta)} .$$

$H^{(2)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, s, \epsilon t)$ is then simply

$$H^{(2)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, s, \epsilon t \cdots) \qquad 2$$
$$= \frac{e_{a_1}}{m_{a_1}}\, g_{a_2}(-\underline{k}, \underline{v}_2) \int \frac{d\underline{\ell}}{(2\pi)^3}\, \frac{(-i(\underline{k}-\underline{\ell}) \cdot \frac{\partial}{\partial \underline{v}_1}\, g_{a_1}(\underline{\ell}, \underline{v}_1)}{s + i(\omega_\ell - \omega_k - \omega_{\ell-k})}\, \psi_{\ell, -k, k-\ell} + \begin{pmatrix} \underline{k} \to -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix} .$$

The contribution of this to $g^{(2)}_{a_1 a_2}$ is given by

$$g^{(2)}_{a_1 a_2}$$
$$= \frac{1}{(2\pi i)^2} \int_{C_1}\int_{C_2} \frac{ds_1\, ds_2}{s - s_1 - s_2}\, \frac{1}{s_1 + L^{(0)}_{a_1}(\underline{k}, \underline{v}_1)}\, \frac{1}{s_2 + L^{(0)}_{a_2}(-\underline{k}, \underline{v}_2)}\, H^{(2)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, s) \qquad 3$$

From Appendix C_2

$$\frac{1}{s_2 + L^{(0)}_{a_2}(-\underline{k}, \underline{v}_2)}\, g_{a_2}(-\underline{k}, \underline{v}_2) = \frac{1}{s_2 - i\omega_{\underline{k}}}\, g_{a_2}(-\underline{k}, \underline{v}_2) .$$

Closing C_2 to the left and then C_1 to the right gives for 3

$$g^{(2)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, s, \epsilon t \cdots)$$

$$= \frac{g_{a_2}(-\underline{k}, \underline{v}_2)}{s - i\omega_k + L^{(0)}_{a_1}(\underline{k}, \underline{v}_1)} \frac{e_{a_1}}{m_{a_1}} \int \frac{d\underline{\ell}}{(2\pi)^3} \frac{-i(\underline{k}-\underline{\ell})\cdot\frac{\partial}{\partial \underline{v}_1} g_{a_1}(\underline{\ell}, \underline{v}_1)\, \Psi_{\underline{\ell}, -\underline{k}, \underline{k}-\underline{\ell}}}{(s + i(\omega_\ell - \omega_k - \omega_{\ell-k}))} + \left(\begin{matrix}\underline{k} \to -\underline{k} \\ 1 \leftrightarrow 2\end{matrix}\right). \qquad 4$$

For calculating the long-time behavior of 4 we note that the first factor $\dfrac{1}{s - i\omega_k + L^{(0)}_{a_1}}$ certainly has a pole at $s = 0$ through the effect of $\dfrac{1}{\epsilon(\underline{k}, s - i\omega_{\underline{k}})}$. The long t inversion of 4 may then be written

$$g^{(2)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, t \to \infty) = \lim_{s \to 0+} s\, g^{(2)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, s)$$

$$= g_{a_1}(\underline{k}, \underline{v}_1)\, g_{a_2}(-\underline{k}, \underline{v}_2) \qquad 5$$

$$\times \left[\sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2}{m_{a_1}} \frac{1}{\left.\frac{\partial \epsilon_k}{\partial s}\right|_{s = -i\omega_k}} \int \frac{d\underline{\ell}}{(2\pi)^3} \frac{\Psi_{\underline{\ell}, -\underline{k}, \underline{k}-\underline{\ell}}}{(\Delta - i(\omega_\ell - \omega_k - \omega_{\ell-k}))} \int \frac{d\underline{v}_1\, (-i(\underline{k}-\underline{\ell})\cdot\frac{\partial}{\partial \underline{v}_1} g_{a_1}(\underline{\ell}, \underline{v}_1))}{(-i\omega_{\underline{k}} + i\underline{k}\cdot\underline{v}_1 + \Delta)} + (\underline{k} \to -\underline{k}) \right].$$

Denoting the square bracket by $\tilde{\Psi}_{\underline{k}}$, we see that the effect of including $h^{(2)}_{a_1 a_2 a_3}(o, \epsilon t \cdots)$ in the analysis is to give a contribution to $g^{(2)}_{a_1 a_2}$ of the form

$$g^{(2)}_{a_1 a_2} = g_{a_1}(\underline{k}, \underline{v}_1)\, g_{a_2}(-\underline{k}, \underline{v}_2)\, \tilde{\Psi}_{\underline{k}}. \qquad 6$$

We recognize that this is identical in form to $g^{(1)}_{a_1 a_2}$ given in Eq. III-46 and can be trivially absorbed into this result. It has no effect on the kinetic equation for the waves (which is manifested as secular behavior as t in $g^{(2)}_{a_1 a_2}$). As regards the equation advancing f_{a_1}, it only leads to terms similar in form to the right-hand side of Eq. III-48.

APPENDIX D

Contribution to $g^{(2)}_{a_1 a_2}$ from the Third and Sixth Terms of Eq. III-62

By changing variables in the $\underline{l}$ integration from $\underline{l}$ to $\underline{k}-\underline{l}$ the third and sixth terms of Eq. III-46 can be written

$$\frac{1}{2}\int_{C_1}\int_{C_2} \frac{ds_1\, ds_2 /(2\pi i)^2}{s(s-s_1-s_2)\,(s_1 + L^{(0)}_{a_1}(\underline{k},\underline{v}_1))(s_2 + L^{(0)}_{a_2}(-\underline{k},\underline{v}_2))}$$

$$\times \int \frac{d\underline{l}}{(2\pi)^3}\, \psi_{\underline{l}}\, \psi_{\underline{k}-\underline{l}}\, G_{a_2}(\underline{l}-\underline{k},-\underline{l},\underline{v}_2,s)\, \frac{e_{a_1}}{m_{a_1}}\Big[i(\underline{k}-\underline{l})\cdot\frac{\partial}{\partial \underline{v}_1} g_{a_1}(\underline{l},\underline{v}_1) \qquad 1$$

$$+ i\underline{l}\cdot\frac{\partial}{\partial \underline{v}_1} g_{a_1}(\underline{k}-\underline{l},\underline{v}_1)\Big] \quad + \quad \begin{pmatrix} \underline{k} \rightarrow -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix}$$

Keep in mind $\mathrm{Re}\, s > \mathrm{Re}\,(s_1+s_2)$ for s_1 on C_1 and s_2 on C_2. C_1 and C_2 are the usual Laplace contours parallel to the $\mathrm{Im}\, s_1$ ($\mathrm{Im}\, s_2$) axis and to the right of singularities in the integrand. For brevity we denote

$$\frac{e_{a_1}}{m_{a_1}}\Big[i(\underline{k}-\underline{l})\cdot\frac{\partial}{\partial \underline{v}_1} g_{a_1}(\underline{l},\underline{v}_1) + i\underline{l}\cdot\frac{\partial}{\partial \underline{v}_1} g_{a_1}(\underline{k}-\underline{l},\underline{v}_1)\Big] = \{\underline{v}_1\}_{a_1,\underline{k}-\underline{l},\underline{l}} \qquad 2$$

Also from Eq. III-58

$$G_{a_2}(\underline{l}-\underline{k},-\underline{l},\underline{v}_2,s) = \frac{1}{(s + i(\omega_{k-l}+\omega_l) + L^{(0)}_{a_2}(-\underline{k},\underline{v}_2))}\{\underline{v}_2\}_{a_2,\underline{l}-\underline{k},-\underline{l}} \qquad 3$$

Performing the s_1 integration in (1) by closing the C_1 contour to the right and picking up the only pole (at $s_1 = s - s_2$) within readily gives for

$$(1),\ \frac{1}{2}\int \frac{d\underline{l}}{(2\pi)^3}\,\psi_{\underline{l}}\,\psi_{\underline{k}-\underline{l}} \int_{C_2} \frac{ds_2/(2\pi i)}{s\,(s_2 + L_{a_2})(s-s_2+L_{a_1})(s+i(\omega_{k-l}+\omega_l)+L_{a_2})}$$

$$\times \{\underline{v}_1\}_{a_1,k-l,l}\ \{\underline{v}_2\}_{a_2\, l-k,-l} \quad + \quad \begin{pmatrix} \underline{k}\rightarrow -\underline{k} \\ 1\leftrightarrow 2 \end{pmatrix}, \qquad 4$$

where we have dropped the arguments of the operators $L^{(0)}_{a_1}(\underline{k},\underline{v}_1)$ and $L^{(0)}_{a_2}(-\underline{k},\underline{v}_2)$ for simplicity of notation. One can show simply that the operator product

$$\frac{1}{(s_2+L_{a_2})}\cdot\frac{1}{(s+i(\omega_{k-l}+\omega_l)+L_{a_2})}\left\{\underline{V}_2\right\}_{a_2,l-k,-l}$$

$$\equiv\frac{1}{s-s_2+i(\omega_{k-l}+\omega_l)}\left[\frac{1}{s_2+L_{a_2}}-\frac{1}{s+i(\omega_{k-l}+\omega_l)+L_{a_2}}\right]\left\{\underline{V}_2\right\}_{a_2,l-k,-l} \quad 5$$

Using (5) in (4), the last portion of (5) gives zero contribution as can be seen by closing C_2 to the left for this term. The operator $\frac{1}{s-s_2+L_{a_1}}$ is analytic for $\mathrm{Re}\,(s-s_2)>0$, i.e., $\mathrm{Re}\;s_2<\mathrm{Re}\;s$, hence it has no poles to the left of C_2 (note: $\mathrm{Re}\,s>\mathrm{Re}\,s_2$ on C_2). Similarly $\frac{1}{s-s_2+i(\omega_{k-l}+\omega_l)}$ has no singularity to the left of C_2; hence the contribution zero from the last term of (5) when substituted into (4). The resulting expression for(4) becomes

$$\frac{1}{(2\pi i)}\frac{1}{2}\int\frac{d\underline{l}}{(2\pi)^3}\psi_{\underline{l}}\psi_{\underline{k}-\underline{l}}\int_{C_2}\frac{ds_2}{s(s-s_2+i(\omega_{k-l}+\omega_l))(s-s_2+L_{a_1})(s_2+L_{a_2})} \quad 6$$
$$\times\left\{\underline{V}_1\right\}_{a_1,k-l,l}\left\{\underline{V}_2\right\}_{a_2,l-k,-l}\qquad+\begin{pmatrix}\underline{k}\to-\underline{k}\\ 1\leftrightarrow 2\end{pmatrix}.$$

(6) may be used as it stands, or if we transform variables from $s_2\to s_2-i(\omega_{k-l}+\omega_l)$ it becomes

$$\frac{1}{(2\pi i)}\frac{1}{2}\int\frac{d\underline{l}}{(2\pi)^3}\psi_l\psi_{k-l}\int_{C_2}\frac{ds_2}{s(s-s_2)(s-s_2-i(\omega_{k-l}+\omega_l)+L_{a_1})(s_2+i(\omega_{k-l}+\omega_l)+L_{a_2})} \quad 7$$
$$\times\left\{\underline{V}_1\right\}_{a_1,k-l,l}\left\{\underline{V}_2\right\}_{a_2,l-k,-l}\qquad+\begin{pmatrix}\underline{k}\to-\underline{k}\\ 1\leftrightarrow 2\end{pmatrix}.$$

We use the result (6) or (7) whichever is most convenient. Before proceeding to calculate the longtime contribution of (6) (or (7)) to $g^{(2)}_{a_1a_2}$, we attack the simpler problem of calculating its contribution to the second order coulomb interaction energy density.

Second-Order Energy Density from the Third and Sixth Terms of Eq. III-62

In reference to Appendix A, the energy density of interest is proportional to

$$\frac{1}{K^4}\sum_{a_1,a_2}(4\pi n_{a_1}e_{a_1})(4\pi n_{a_2}e_{a_2})\iint d\underline{v}_1\, d\underline{v}_2\; g^{(2)}_{a_1 a_2}\ . \qquad 8$$

Utilizing the identity

$$\sum_{a_1} 4\pi n_{a_1}e_{a_1}\int d\underline{v}_1 \frac{1}{(s+L^{(0)}_{a_1}(\underline{k},\underline{v}_1))} = \frac{1}{\epsilon(\underline{k},s)}\sum_{a_1}(4\pi n_{a_1}e_{a_1})\int\frac{d\underline{v}_1}{s+i\underline{k}\cdot\underline{v}_1}\ ,$$

and that portion of $g^{(2)}_{a_1 a_2}$ given by (6), (8) becomes

$$\frac{1}{2}\frac{1}{(2\pi i)}\int\frac{d\underline{l}}{(2\pi)^3}\psi_{\underline{l}}\,\psi_{\underline{k}-\underline{l}}\int_{C_2}\frac{ds_2}{s(s-s_2+i(\omega_{k-l}+\omega_l)\quad \epsilon(k,s-s_2)\ \epsilon(-\underline{k},s_2)}$$

$$\times\frac{1}{k^2}\int\frac{dv_1}{(s-s_2+ik\cdot v_1)}\Big(\sum_{a_1}(4\pi n_{a_1}e_{a_1})\{\underline{v}_1\}_{a_1,k-l,l}\Big)\frac{1}{k^2}\int\frac{dv_2}{s_2-ik\cdot v_2}\Big(\sum_{a_2}4\pi n_{a_2}e_{a_2}\{v_2\}_{a_2,l-k,-l}\Big) \qquad 9$$

$$+\quad(\underline{k}\to-\underline{k})\ .$$

We denote (9) as $E_k^{(2)}(s)$. The point is that the inversion of (9) leads to secular behavior as t. We demonstrate this in two different ways. Although the first method leaves much to be desired as for rigour it serves to exhibit the secularity.

Method 1

Recall we have assumed that $\epsilon(k,s)=0$ yields a single marginally stable mode $s_k=-i\omega_k$. There will also be zeros corresponding to Landau damping (zeros in the left-hand plane of $\epsilon(k,s)$). We have been careful thus far to close the various contours away from the region of damping, that is, close the contours around regions where the analytic properties of the different operators are well known.

However in reference to (9) if we temporarily assume that the only zero of $\epsilon(-k, s_2)$ is for $s_2 = i\omega_k$ and that we can close C_2 to the left, picking up only this pole, the resulting expression clearly has a double pole at $s=0$ through $\frac{1}{s\,\epsilon(k, s-i\omega_k)}$. $\lim_{s\to 0+} s E_k^{(2)}(s)$ then varies as $1/s$, giving for large t,

$$E_k^{(2)}(t) \simeq \frac{t}{2}\int \frac{d\underline{l}}{(2\pi)^3} \frac{\psi_l \psi_{k-l}}{\left|\frac{\partial \epsilon_k}{\partial s}\right|^2_{s=-i\omega_k}} \frac{1}{\Delta - i(\omega_k - \omega_l - \omega_{k-l})} \tag{10}$$

$$\times \left| \bar{\mu}(\underline{k}-\underline{l}, \omega_{k-l}; \underline{l}, \omega_l; \underline{k}, \omega_k)\right|^2 + (\underline{k} \to -\underline{k}),$$

where

$$\bar{\mu}(\underline{k}-\underline{l}, \omega_{k-l}; \underline{l}, \omega_l; \underline{k}, \omega_k) = \sum_a \frac{(4\pi n_a e_a)}{k^2} \int \frac{d\underline{v}}{\Delta - i\omega_k + i\underline{k}\cdot\underline{v}} \{\underline{v}\}_{a, k-l, l}. \tag{11}$$

The principal part in (10) vanishes because of antisymmetry when $\underline{k} \to -\underline{k}$ and integration variables are changed from $\underline{l} \to -\underline{l}$, thus giving for (10)

$$E_k^{(2)}(t) \simeq t\,\frac{\pi}{2} \frac{1}{\left|\frac{\partial \epsilon_k}{\partial s}\right|^2_{s=-i\omega_k}} \int \frac{d\underline{l}}{(2\pi)^3} \psi_l \psi_{k-l}\, \delta(\omega_k - \omega_l - \omega_{k-l}) \left|\bar{\mu}(k-l, \omega_{k-l}; l, \omega_l; k, \omega_k)\right|^2 + (\underline{k} \to -\underline{k}).$$

Method 2

In this approach we push C_2 up to the imaginary s_2 axis, changing variables to $s_2 = -i\omega + \Delta$. As $\mathrm{Re}\, s > \mathrm{Re}\, s_2$ for s_2 on C_2, we write $s = 2\Delta$ and to find the large t behavior consider $s \to 0+$. We expect to find behavior as $1/s^2$ leading to a secularity as t.

(9) becomes

$$\frac{1}{2s}\frac{1}{(2\pi i)}\int \frac{d\underline{l}}{(2\pi)^3} \psi_l \psi_{k-l} \int_{-\infty}^{\infty} \frac{i\, d\omega}{\Delta + i(\omega + \omega_{k-l} + \omega_l)} \frac{1}{\epsilon(k, \Delta + i\omega)} \frac{1}{\epsilon(-k, \Delta - i\omega)} \tag{12}$$

$$\times \frac{1}{k^2}\int \frac{dv_1}{\Delta + i\omega + ik\cdot v_1}\left(\sum_{a_1} (4\pi n_{a_1} e_{a_1})\{v_1\}_{a_1, k-l, l}\right) \frac{1}{k^2}\int \frac{dv_2}{\Delta - i\omega - ik\cdot v_2}\left(\sum (4\pi n_{a_2} e_{a_2})\{v_2\}_{a_2, l-k, -l}\right)$$

$$+ (\underline{k} \to -\underline{k})$$

The $\dfrac{P}{\omega+\omega_{k-\ell}+\omega_\ell}$ term vanishes because of antisymmetry then giving

$$\frac{1}{2s}\int\frac{d\underline{\ell}}{(2\pi)^3}\,\psi_\ell\,\psi_{k-\ell}\,\frac{1}{2}\,\frac{1}{\epsilon(\underline{k},\,\Delta-i(\omega_{k-\ell}+\omega_\ell))}\;\frac{1}{\epsilon(-\underline{k},\Delta+i(\omega_{k-\ell}+\omega_\ell))} \qquad 13$$

$$\times\,|\bar{\mu}(\underline{k}-\underline{\ell},\omega_{k-\ell};\,\ell,\omega_\ell;\,k,\,\omega_{k-\ell}+\omega_\ell)|^2 \qquad + (\underline{k}\to-\underline{k})\,.$$

If for a given $\underline{k}$ there exists $\underline{\ell}$ such that $\omega(\underline{k})=\omega(\underline{k}-\underline{\ell})+\omega(\underline{\ell})$ it is clear that (13) is singular, for if we were to write (as in Ref. 7)

$$\frac{1}{\epsilon(\underline{k},\,\Delta-i(\omega_{k-\ell}+\omega_\ell))}=\frac{P}{\epsilon(\underline{k},\,-i(\omega_{k-\ell}+\omega_\ell))}+\pi\,\frac{\delta(\omega_k-\omega_\ell-\omega_{k-\ell})}{\left.\frac{\partial\epsilon_k}{\partial s}\right|_{s=-i\omega_k}}, \qquad 14$$

then part of the integrand in (13) involves the product of two $\delta(\omega_k-\omega_\ell-\omega_{k-\ell})$ functions. This is just a reflection of the fact that that part of the energy density varies as $1/s^2$ and really leads to a secular behavior as t. Similar terms are encountered in Ref. 15 in the consideration of wave-wave interactions in a simpler problem. The prescription here, as there, is

$$\frac{\pi}{2}^2\,\delta(x)\,\delta(x)\;\longrightarrow\;\frac{\pi\,\delta(x)}{s} \qquad 15$$

i.e.

$$\frac{\pi}{2}^2\,\delta(x)\delta(x)=\frac{\pi}{2}^2\,\delta(x)\,\frac{\Delta}{\pi(x^2+\Delta^2)}\equiv\frac{\pi\,\delta(x)}{2\Delta}\equiv\frac{\pi\,\delta(x)}{s} \qquad 16$$

The secular part of (13) is then trivially

$$t \left\{ \frac{\pi}{2} \int \frac{d\ell}{(2\pi)^3} \frac{\psi_\ell \psi_{k-\ell}}{\left|\frac{\partial \mathcal{E}_k}{\partial s}\right|^2_{s=-i\omega_k}} \delta(\omega_k - \omega_\ell - \omega_{k-\ell}) \left| \bar{\mu}(k-\ell, \omega_{k-\ell}; \ell, \omega_\ell; k, \omega_k) \right|^2 + (k \to -k) \right\}, \qquad 17$$

as in (10). Thus that portion of $g^{(2)}_{a_1 a_2}$ given by the third and sixth terms of Eq. III-62 leads to secular behavior in the second order energy density of coulomb interaction. In Appendix E we calculate the remainder of the energy density from the rest of the terms in III-62 and III-63. Removing secular behavior will then give us a kinetic equation for the waves.

As discussed elsewhere (Appendix H_2) the validity of (10)or (13) is contingent on the assumption that

$$\omega(k) - \omega(\ell) = \omega(k-\ell) \quad \text{and} \quad \frac{d}{d\ell}\left(\omega(\ell) + \omega(k-\ell)\right) = 0 \qquad 18$$

cannot be satisfied simultaneously for some ℓ (given k). We now return to the problem of finding the contribution of the third and sixth terms of III-62 to $g^{(2)}_{a_1 a_2}$.

<u>Solution for $g^{(2)}_{a_1 a_2}$</u>

For convenience denote (7), by g(s). We have in mind following the same procedure in "Method 2", namely writing $s_2 = -i\omega + \Delta$, $s = 2\Delta$ and obtaining the long t behavior from $\lim_{s \to 0+} s\, q(s)$. (7) may then be written

$$s\, q(s) = \frac{1}{2} \int \frac{d\ell}{(2\pi)^3} \psi_\ell \psi_{k-\ell} \frac{1}{(2\pi)} \int_{-\infty}^{\infty} d\omega \frac{1}{\Delta + i\omega} \frac{1}{(\Delta + i\omega - i(\omega_{k-\ell} + \omega_\ell) + L^{(0)}_{a_1}(k, v_1))} \qquad 19$$

$$\times \frac{1}{-i\omega + \Delta + i(\omega_{k-\ell} + \omega_\ell) + L^{(0)}_{a_2}(-k, v_2)} \times \{v_1\}_{a_1, k-\ell, \ell} \{v_2\}_{a_2, \ell-k, -\ell} + \begin{pmatrix} k \to -k \\ 1 \leftrightarrow 2 \end{pmatrix}$$

The P/ω term vanishes because of antisymmetry in ω leaving the contribution from the $\delta(\omega)$, namely

$$\frac{1}{4}\int\frac{d\underline{\ell}}{(2\pi)^3}\psi_\ell\psi_{k-\ell}\left[\frac{1}{(\Delta-i(\omega_{k-\ell}+\omega_\ell)+L^0_{a_1}(\underline{k},\underline{v}_1))}\{\underline{v}_1\}_{a_1,\underline{k}-\underline{\ell},\underline{\ell}}\right.$$
$$\left.\times\frac{1}{(\Delta+i(\omega_{k-\ell}+\omega_\ell)+L^0_{a_2}(-\underline{k},\underline{v}_2))}\{\underline{v}_2\}_{a_2,\ell-k,-\ell}\right]+\binom{\underline{k}\to-\underline{k}}{1\leftrightarrow 2}. \qquad 20$$

Careful examination of the $\frac{1}{S+L}$ operators in (20) reveals a term proportional to the product of two $\delta(\omega_k-\omega_\ell-\omega_{k-\ell})$ functions and identical in form to the one previously discussed. Recognizing this as a manifestation of secular behavior, we can write (20) for large t as

$$\left(\frac{1}{4}\int\frac{d\underline{\ell}}{(2\pi)^3}\psi_\ell\psi_{k-\ell}\,\eta^{a_1a_2}_{\underline{k}-\underline{\ell},\underline{\ell}}+\binom{\underline{k}\to-\underline{k}}{1\leftrightarrow 2}\right)+t\,g_{a_1}(\underline{k},\underline{v}_1)\,g_{a_2}(-\underline{k},\underline{v}_2)\{\ \},\qquad 21$$

where $\{\ \}$ is the curly bracket of (17) and $\eta^{a_1a_2}_{\underline{k}-\underline{\ell},\underline{\ell}}$ denotes everything which occurs within the square bracket of Eq. 20 except that term corresponding to the product of two $\delta(\omega_k-\omega_\ell-\omega_{k-\ell})$ functions. The first part of (21) is the contribution of the third and sixth terms of Eq. III-62 to $g^{(2)}_{a_1a_2}(t\to\infty)$ to be used in Eq. II-34 to determine the evolution of the first distribution. The secular part of (21), when grouped with the remaining secular parts of $g^{(2)}_{a_1a_2}$ from Eq. III-62 and Eq. III-63 will lead to a kinetic equation for ψ_k on the ϵt scale. The resulting equation will be identical to that obtained by asking that the net second-order energy density of coulomb interaction be given by a uniformly valid expression.

APPENDIX E

Calculation of $g^{(2)}_{a_1 a_2}$

In this section we examine that part of $g^{(2)}_{a_1 a_2}$ given by expression III-63 and the first two terms of III-62 (together with their $\left(\begin{smallmatrix} \underline{k} \to -\underline{k} \\ 1 \leftrightarrow 2 \end{smallmatrix}\right)$ versions). The effect of the third and the sixth terms in III-63 are considered in Appendix D . We preface a discussion of the explicit form of $g^{(2)}_{a_1 a_2}$ by looking at a physical quantity of considerable interest, namely the energy density of coulomb interaction (see Appendix A).

Energy Density of Coulomb Interaction

The quantity of interest is proportional to

$$\frac{1}{K^4} \sum_{a_1, a_2} (4\pi n_{a_1} e_{a_1})(4\pi n_{a_2} e_{a_2}) \iint d\underline{v}_1\, d\underline{v}_2\, g^{(2)}_{a_1, a_2} . \qquad 1$$

Utilizing the identity

$$\sum_{a_1} (4\pi n_{a_1} e_{a_1}) \int d\underline{v} \frac{1}{s + L^{(0)}_{a_1}(\underline{k}, \underline{v})} \equiv \frac{1}{\epsilon(\underline{k}, s)} \sum_{a_1} (4\pi n_{a_1} e_{a_1}) \int \frac{d\underline{v}}{s + i\underline{k}\cdot\underline{v}} \qquad 2$$

and noting that

$$\frac{1}{k^2} \sum_{a_2} (4\pi n_{a_2} e_{a_2}) \int d\underline{v}_2\, g_{a_2}(-\underline{k}, \underline{v}_2) = 1 , \qquad 3$$

(1) becomes (except for the third and sixth terms in III-62)

$$\Psi_k^{(2)}(s) = \frac{1}{s\, \epsilon(k, s - i\omega_k)} \left\{ \int \frac{d\underline{\ell}}{(2\pi)^3} \Psi_k \Psi_{k-\ell} \sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2}{m_{a_1} k^2} \int \frac{d\underline{v}_1}{(s - i\omega_k + ik.v_1)} i(\underline{k}-\underline{\ell})\cdot\frac{\partial}{\partial \underline{v}_1} G(\underline{k}, \underline{\ell} - \underline{k}, v_1, s) \right.$$

$$+ \int \frac{d\underline{\ell}}{(2\pi)^3} \Psi_\ell \Psi_k \sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2}{m_{a_1} k^2} \int \frac{d v_1}{s - i\omega_k + ik.v_1} i(\underline{k}-\underline{\ell})\cdot\frac{\partial}{\partial \underline{v}_1} g_{a_1}(\underline{\ell}, \underline{v}_1)$$

$$\left. \times \sum_{a_3} \frac{4\pi n_{a_3} e_{a_3}}{|\underline{k} - \underline{\ell}|^2} \int d\underline{v}_3\, G_{a_3}(-\underline{\ell}, \underline{k}, \underline{v}_3, s) \right\} + (\underline{k} \to -\underline{k})$$

(cont'd)

$$+\frac{1}{s\,\epsilon(k,s-i\omega_k)}\left\{-\sum_{a_1}\frac{4\pi n_{a_1}e_{a_1}}{k^2}\frac{1}{2}\int\frac{d\underline{v}_1\, g_{a_1}(\underline{k},\underline{v}_1)}{s-i\omega_k+ik.v_1}\frac{\partial\psi_{\underline{k}}}{\partial\epsilon t}\right.$$

$$-\psi_{\underline{k}}\sum_{a_1}\frac{4\pi n_{a_1}e_{a_1}}{k^2}\int d\underline{v}_1\frac{1}{s-i\omega_k+ik.v_1}\frac{\partial}{\partial\epsilon t}g_{a_1}(\underline{k},\underline{v}_1) \qquad 4$$

$$\left.+\psi_{\underline{k}}\sum_{a_1}\frac{4\pi n_{a_1}e_{a_1}^2}{m_{a_1}k^2}\int d\underline{v}_1\frac{i\underline{k}\cdot\partial f_{a_1}^{(1)}/\partial\underline{v}_1}{(s-i\omega_k+ik.v_1)}\right\}$$

$$+(\underline{k}\rightarrow-\underline{k}).$$

For large t, the double pole at $s=0$ renders (4) secular as t. It is convenient at this point to record some identities useful in considering (4) and introduce some notation.

Recalling the definition of G as given in Eq. III-58, we note that

$$\sum_{a_3}(4\pi n_{a_3}e_{a_3})\int d\underline{v}_3\, G_{a_3}(-\underline{\ell},\underline{k},\underline{v}_3,s)$$

$$=\frac{1}{\epsilon(\underline{k}-\underline{\ell},-i(\omega_k-\omega_\ell)+s)}\sum_{a_3}\frac{4\pi n_{a_3}e_{a_3}^2}{m_{a_3}}$$

$$\times\int\frac{dv_3}{(s-i(\omega_k-\omega_\ell)+i(k-\ell)\cdot\underline{v})}\left\{i\underline{k}\cdot\frac{\partial}{\partial\underline{v}_3}g_{a_3}(-\underline{\ell},\underline{v}_3)-i\underline{\ell}\cdot\frac{\partial}{\partial\underline{v}_3}g_{a_3}(\underline{k},\underline{v}_3)\right\} \qquad 5$$

Also

$$\frac{1}{\epsilon(k,s-i\omega_k)}\sum_{a_1}\frac{(4\pi n_{a_1}e_{a_1})}{k^2}\int\frac{d\underline{v}_1\; g_{a_1}(\underline{k},\underline{v}_1)}{s-i\omega_k+ik\cdot v_1}=\frac{1}{s}$$

Keeping in mind we wish to find the long time behavior of $\psi_k^{(2)}$ by considering $\lim_{s\to 0+} s\,\psi_k^{(2)}(s)$, it is convenient to introduce the notation

$$\bar{\mu}(\underline{k}_1,\omega_1;\underline{k}_2,\omega_2;\underline{k}_3,\omega_3)$$

$$= \int d\underline{v}\; \bar{\mu}_v(k_1,\omega_1;k_2,\omega_2;k_3,\omega_3)$$

$$= \mu(k_1,\omega_1;k_2,\omega_2;k_3,\omega_3) + \mu(k_2,\omega_2;k_1,\omega_1;k_3,\omega_3)$$

$$= \int d\underline{v}\,\big(\mu_v(k_1,\omega_1;k_2,\omega_2;k_3,\omega_3) + \mu_v(k_2,\omega_2;k_1,\omega_1;k_3,\omega_3)\big) \tag{6}$$

$$= \sum_{a_1}\int d\underline{v}\,\big(\mu_{\underline{v}}^{a_1}(k_1,\omega_1;\underline{k}_2,\omega_2;k_3,\omega_3) + \mu_{\underline{v}}^{a_1}(\underline{k}_2,\omega_2;k_1,\omega_1;k_3,\omega_3)\big)$$

$$\equiv \sum_{a_1}\frac{4\pi n_{a_1}e_{a_1}^3}{k_3^2\, m_{a_1}^2}\int\frac{d\underline{v}}{(\Delta - i\omega_3 + i\underline{k}_3\cdot\underline{v})}\left\{ i\underline{k}_1\cdot\frac{\partial}{\partial\underline{v}_1}\;\frac{i\underline{k}_2\cdot\partial f_{a_1}^{0}/\partial v}{\Delta - i\omega_2 + i\underline{k}_2\cdot\underline{v}}\right.$$
$$\left. + i\,\underline{k}_2\cdot\frac{\partial}{\partial\underline{v}}\;\frac{i\underline{k}_1\cdot\partial f_{a_1}^{0}/\partial\underline{v}}{\Delta - i\omega_1 + i\underline{k}_1\cdot\underline{v}}\right\},$$

where the definitions are self-evident. With these points in mind, we consider

$$\lim_{s\to 0+} s\,\psi_k^{(2)}(s)\;. \tag{7}$$

This behaves as $1/s$ because of the double pole at $s = 0$, leading to a secularity as t. After some tedious algebraic manipulation, for example changing variables from $\underline{l}$ to $\underline{k}-\underline{l}$ in certain $\underline{l}$ integrals, and using the symmetry $\psi_{\underline{k}} = \psi_{-\underline{k}}$, then for large t, (4) becomes

$$+ \left\{ \frac{1}{\left.\frac{\partial \mathcal{E}_k}{\partial s}\right|_{s=-i\omega_k}} \int \frac{d\underline{l}}{(2\pi)^3} \Psi_{\underline{k}} \Psi_{\underline{l}} \left[\frac{\bar{\mu}(\underline{k},\omega_k; -\underline{l},\omega_{-l}; \underline{k}-\underline{l}, \omega_k-\omega_l)}{\epsilon(\underline{k}-\underline{l}, -i(\omega_k-\omega_l)+\Delta)} \right.\right.$$

$$\times \quad \bar{\mu}(\underline{k}-\underline{l}, \omega_k-\omega_l; \underline{l}, \omega_l; \underline{k}, \omega_k)$$

$$\left. + \sum_{a_1} \frac{e_{a_1}}{m_{a_1}} \frac{|\underline{k}-\underline{l}|^2}{K^2} \int \frac{d\underline{v}}{(-i\omega_k + i\underline{k}\cdot\underline{v}+\Delta)} i\underline{l}\cdot\frac{\partial}{\partial \underline{v}} \bar{\mu}^{a_1}_{\underline{v}}(\underline{k},\omega_k; -\underline{l},\omega_{-l}; \underline{k}-\underline{l}\, \omega^8_{k-l}) \right]$$

$$- \frac{1}{2}\frac{\partial \Psi_{\underline{k}}}{\partial \epsilon t} - \Psi_{\underline{k}} \sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2}{m_{a_1} K^2 \left.\frac{\partial \mathcal{E}_k}{\partial s}\right|_{s=-i\omega_k}} \int \frac{d\underline{v}_1}{\Delta - i\omega_k + i\underline{k}\cdot\underline{v}_1} \frac{\partial}{\partial \epsilon t} \frac{i\underline{k}\cdot\partial f^0_{a_1}/\partial \underline{v}_1}{(\Delta - i\omega_k + i\underline{k}\cdot\underline{v}_1)}$$

$$\left. + \Psi_{\underline{k}} \sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2/m_{a_1}}{K^2 \left.\frac{\partial \mathcal{E}_k}{\partial s}\right|_{s=-i\omega_k}} \int d\underline{v}_1 \frac{i\underline{k}\cdot\partial f^{(1)}_{a_1}/\partial \underline{v}_1}{(\Delta - i\omega_k + i\underline{k}\cdot\underline{v}_1)} \right\} + t\left\{ \underline{k} \to -\underline{k} \right\} .$$

The net energy density of coulomb interaction associated with $g^{(2)}_{a_1 a_2}$ (apart from its initial value $g^{(2)}_{a_1 a_2}(\underline{k}, \underline{v}_1, \underline{v}_2, 0, \epsilon t\cdot)$) consists of adding (8) and Eq. (17) of Appendix D . In order that the energy density be given by a uniformly valid expression we set the net coefficient equal to zero thus giving a kinetic equation for $\Psi_{\underline{k}}$ on the ϵt scale; the $(\underline{k} \to -\underline{k})$ terms add to the existing terms their complex conjugate.

We thus have

$$\frac{\partial \Psi_k}{\partial \epsilon t} = 2\gamma_k \Psi_k - 2\Psi_k \, Re\left(\sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2}{m_{a_1} k^2 \left.\frac{\partial \mathcal{E}_k}{\partial s}\right|_{s=-i\omega_k}} \int d\underline{v} \frac{1}{\Delta - i\omega_k + ik\cdot v} \frac{\partial}{\partial \epsilon t} \frac{ik\cdot \partial f_{a_1}^{(0)}/\partial v}{\Delta - i\omega_k + ik\cdot v} \right)$$

$$+ 2\Psi_k \, Re\left(\frac{1}{\left.\frac{\partial \mathcal{E}_k}{\partial s}\right|_{s=-i\omega_k}} \int \frac{d\underline{\ell}}{(2\pi)^3} \Psi_\ell \left[\frac{\bar{\mu}(\underline{k},\omega_k; -\underline{\ell}, \omega_{-\ell}; \underline{k}-\underline{\ell}, \omega_k - \omega_\ell)}{\epsilon(\underline{k}-\underline{\ell}, -i(\omega_k - \omega_\ell) + \Delta)} \right.\right. \qquad 9$$

$$\times \; \bar{\mu}(\underline{k}-\underline{\ell}, \omega_k - \omega_\ell; \underline{\ell}, \omega_\ell; \underline{k}, \omega_k)$$

$$\left.\left. + \sum_{a_1} \frac{e_{a_1}}{m_{a_1}} \frac{|\underline{k}-\underline{\ell}|^2}{k^2} \int \frac{d\underline{v}}{(\Delta - i\omega_k + i\underline{k}\cdot\underline{v})} \, i\underline{\ell}\cdot\frac{\partial}{\partial \underline{v}} \, \bar{\mu}_{\underline{v}}^{a_1}(\underline{k},\omega_k; -\underline{\ell}, \omega_{-\ell}; \underline{k}-\underline{\ell}, \omega_k - \omega_\ell) \right]\right)$$

$$+ \frac{\pi}{\left|\frac{\partial \mathcal{E}_k}{\partial s}\right|^2_{s=-i\omega_k}} \int \frac{d\underline{\ell}}{(2\pi)^3} \Psi_\ell \Psi_{k-\ell} \left| \bar{\mu}(k-\ell, \omega_{k-\ell}; \ell, \omega_\ell; k, \omega_k)\right|^2 \delta(\omega_k - \omega_\ell - \omega_{k-\ell}),$$

which is the required equation, where

$$\gamma_k = Re\left(\sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2 / m_{a_1}}{k^2 \; \partial\mathcal{E}_k/\partial s|_{s=-i\omega_k}} \int d\underline{v} \; \frac{ik\cdot \partial f_{a_1}^{(1)}/\partial \underline{v}}{(\Delta - i\omega_k + ik\cdot v)} \right).$$

We emphasize that the contribution of $g^{(2)}_{a_1a_2}(0,\epsilon t)$ to $g^{(2)}_{a_1a_2}$ leads to a term identical in form to $g^{(1)}_{a_1a_2}$ and can be trivially absorbed into the same.

<u>Evaluation of $g^{(2)}_{a_1a_2}$</u>

That part of $g^{(2)}_{a_1a_2}$ from III-62 and III-63 excluding the third and sixth terms of III-62 involves the operator,

$$\frac{1}{s\,(s - i\omega_k + L_{a_1}^{(1)}(\underline{k},\underline{v}_1))} \qquad 10$$

$$= \frac{1}{s(s - i\omega_k + ik\cdot v)} + \frac{e_{a_1}}{m_{a_1}} \frac{i\underline{k}\cdot \partial f^0_{a_1}/\partial \underline{v}_1}{s - i\omega_k + i\underline{k}\cdot \underline{v}_1} \sum \frac{4\pi n_{a_1} e_{a_1}}{k^2 \, s \, \epsilon(k, s - i\omega_k)} \int \frac{d\underline{v}_1}{s - i\omega_k + ik\cdot v_1}$$

as well as its $\begin{pmatrix} k \to -k \\ 1 \leftrightarrow 2 \end{pmatrix}$ version. Because of the double pole at $s = 0$ in the last portion of this operator there is secular behavior as t in $g^{(2)}_{a_1 a_2}$. Careful examination as $s \to 0+$ in fact shows that the secular term is just

$$t\, g_{a_1}(\underline{k}, \underline{v}_1)\, g_{a_2}(-\underline{k}, \underline{v}_2) \left\{ \quad \right\} + \begin{pmatrix} \underline{k} \to -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix} \qquad 11$$

where $\left\{ \quad \right\}$ is the curly bracket of (8).

If we group (11) with the secular part of $g^{(2)}_{a_1 a_2}$ given in Eq. 21, Appendix D (coming from the third and sixth terms of III-62), the same kinetic equation for ψ_k on the ϵt scale results as is given in (9) which was obtained by asking that the energy density of coulomb interaction be uniformly valid.

The nonsecular $t \to \infty$ portion of $g^{(2)}_{a_1 a_2}$ thus arises from the first part of operator (10). In the context of the notation introduced, we have

$$\begin{aligned} g^{(2)}_{a_1 a_2}(t \to \infty) &= \lim_{s \to 0+} s\, g^{(2)}_{a_1 a_2}(s) \\ &= g_{a_2}(-\underline{k}, \underline{v}_2)\, \zeta^{a_1}_{\underline{k}}(\underline{v}_1) + \begin{pmatrix} \underline{k} \to -\underline{k} \\ 1 \leftrightarrow 2 \end{pmatrix} \end{aligned} \qquad 12$$

where

$$\begin{aligned} \zeta^{a_1}_{\underline{k}}(\underline{v}_1) = \Bigg\{ & \frac{k^2}{(4\pi n_{a_1} e_{a_1})} \int \frac{d\underline{\ell}}{(2\pi)^3}\, \psi_k \psi_\ell \Bigg[\frac{\bar{\mu}(\underline{k}, \omega_k; -\underline{\ell}, \omega_{-\ell}; k-\ell, \omega_k - \omega_\ell)}{\epsilon(\underline{k} - \underline{\ell}, -i(\omega_k - \omega_\ell) + \Delta)} \\ & \times \bar{\mu}^{a_1}_{v_1}(\underline{k} - \underline{\ell}, \omega_k - \omega_\ell; \ell, \omega_\ell; k, \omega_k) \\ & + \frac{e_{a_1}}{m_{a_1}} \frac{|\underline{k} - \underline{\ell}|^2}{k^2} \frac{1}{(-i\omega_k + i\underline{k}\cdot\underline{v}_1 + \Delta)}\, i\underline{\ell} \cdot \frac{\partial}{\partial \underline{v}_1} \bar{\mu}^{a_1}_{v_1}(\underline{k}, \omega_k; -\ell, \omega_{-\ell}; k-\ell, \omega_k - \omega_\ell) \Bigg] \\ & - \frac{g_{a_1}(\underline{k}, \underline{v}_1)}{(\Delta - i\omega_k + i\underline{k}\cdot\underline{v}_1)} \frac{1}{2} \frac{\partial \psi_k}{\partial \epsilon t} - \frac{\psi_k}{(\Delta - i\omega_k + i\underline{k}\cdot\underline{v}_1)} \frac{\partial}{\partial \epsilon t} g_{a_1}(\underline{k}, \underline{v}_1) \\ & + \psi_{\underline{k}} \frac{e_{a_1}}{m_{a_1}} \frac{i\underline{k} \cdot \partial f^{(1)}_{a_1} / \partial \underline{v}_1}{(\Delta - i\omega_k + i\underline{k}\cdot\underline{v}_1)} \Bigg\} . \end{aligned} \qquad 13$$

The net expression for $g^{(2)}_{a_1 a_2}$ to be used in Eq. II-34 to determine the change of the first distribution is just (12) plus the nonsecular contribution from the third and sixth terms of III-62 given by the first term of Eq. 21, Appendix D .

APPENDIX F

Calculation of Collisional Source Term S_k for the Kinetic Equation for the Waves

Noting that $$\sum_{a_1} \frac{4\pi n_{a_1} e_{a_1}^2}{m_{a_1} k^2} \int \frac{i\underline{k}\cdot \partial f_{a_1}^{(0)}/\partial \underline{v}_1}{s_1 + i\underline{k}\cdot\underline{v}_1} = 1 - \epsilon(\underline{k}, s_1),$$

Eq.IV-7 becomes

$$\frac{4\pi}{k^4} \int_{C_1}\int_{C_2} \frac{ds_1 ds_2/(2\pi i)^2}{s(s-s_1-s_2)} \frac{1-\epsilon(\underline{k},s_1)}{\epsilon(\underline{k},s_1)} \frac{1}{\epsilon(-\underline{k},s_2)} \sum 4\pi n_{a_1} e_{a_1}^2 \int \frac{d\underline{v}_2 f_{a_2}^{(0)}}{s_2 - i\underline{k}\cdot\underline{v}_2} \qquad 1$$
$$+ (\underline{k} \to -\underline{k})$$

The portion of this corresponding to $\dfrac{\epsilon(\underline{k}\, s_1)}{\epsilon(\underline{k}, s_1)}$,

gives zero upon closing C_1 to the right and then C_2 to the right, leaving from the $1/\epsilon(\underline{k}, s_1)$ term

$$\frac{4\pi}{k^4} \int_{C_1}\int_{C_2} \frac{ds_1 ds_2/(2\pi i)^2}{s(s-s_1-s_2)} \frac{1}{\epsilon(\underline{k},s_1)\epsilon(-\underline{k},s_2)} \sum_a 4\pi n_a e_a^2 \int \frac{d\underline{v}\, f_a^{(0)}}{s_2 - i\underline{k}\cdot\underline{v}} \cdot \qquad 2$$
$$+ (\underline{k} \to -\underline{k})$$

Denoting 2 by F(s), we see that 2 is just a statement in Laplace variables that

$$\frac{\partial F(t)}{\partial t} = \frac{4\pi}{k^4} \frac{1}{(2\pi i)^2} \int_{C_1}\int_{C_2} \frac{ds_1 ds_2\, e^{(s_1+s_2)t}}{\epsilon(\underline{k},s_1)\epsilon(-\underline{k},s_2)} \sum_a 4\pi n_a e_a^2 \int \frac{d\underline{v}\, f_a^{(0)}}{s_2 - i\underline{k}\cdot\underline{v}_2} \cdot \qquad 3$$
$$+ (\underline{k} \to -\underline{k})$$

If we pull the Laplace contours C_1 and C_2 over to the left picking up the poles $s_1 = -i\omega_{\underline{k}}$; $s_2 = i\omega_{\underline{k}}$, then for large t, assuming the contribution along C_1 and C_2 vanishes as $t \to \infty$, we have

$$\frac{\partial F}{\partial t} = \frac{4\pi}{k^4} \times \frac{1}{\left|\frac{\partial \epsilon_k}{\partial s}\right|^2_{s=-i\omega_k}} \times \sum_a 4\pi n_a e_a^2 \int \frac{d\underline{v}\, f_a^{(0)}}{\Delta + i\omega_k - i\underline{k}\cdot\underline{v}} \qquad 4$$
$$+ (\underline{k} \to -\underline{k}),$$

i.e. $$F(t) \to t\, \mathcal{S}_{\underline{k}}\,. \qquad 5$$

which is the required result.

APPENDIX G

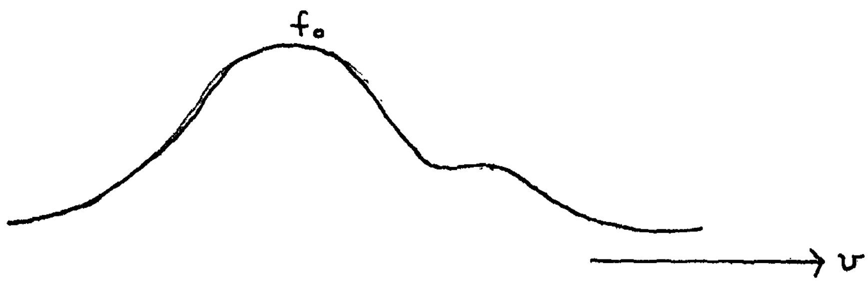
Quasi-Particles

In this section we indicate by simple arguments that the quantity

$$\frac{|\hat{E}_\kappa|^2}{8\pi}\,\omega_\kappa\,\frac{\partial \mathcal{E}(\kappa,\omega)}{\partial\omega}\bigg|_{\omega_\kappa} \qquad 1$$

is a meaningful entity to consider when discussing problems in which the electric field amplitude E_k is subject to slow variations in time. It represents the field energy in the κ-th mode, $|\hat{E}_\kappa|^2/8\pi$, plus the polarization energy due to the interaction of the perturbed particles with the electric field.

We take a particularly simple model of a one-dimensional Vlasov plasma supporting a band of marginally stable modes where the background distribution is of the form



Phenomena corresponding to κ outside the region of marginally stable modes will be Landau damped and we do not include them here. Apart from streaming terms, the solution for large t of the linearized Vlasov equation

$$\frac{\partial f_\kappa}{\partial t} + i\kappa v f_\kappa = \frac{e}{m} E_\kappa \frac{\partial f_o}{\partial v} \qquad 2$$

is

$$f_\kappa = \frac{ie}{m}\,\frac{\hat{E}_\kappa}{|\kappa|}\,\frac{\kappa\,\partial f^o/\partial v}{\omega_\kappa - \kappa v}\,e^{-i\omega_\kappa t}, \qquad 3$$

$$E_\kappa = \frac{\kappa}{|\kappa|}\,\hat{E}_\kappa\,e^{-i\omega_\kappa t}, \qquad 4$$

where ω_k satisfies

$$1 + \frac{\omega_p^2}{k^2}\int \frac{k\,\partial f_o/\partial v}{\omega_k - kv}\,dv = 0 \qquad 5$$

All quantities are well defined as $\partial fo/\partial v = 0$ for those $v = \omega_k/k$. The above

model is exceedingly simple for there is no instability, norinclusion of mode coupling terms. However it serves to illustrate our point, namely that in problems where E_k is subject to slow variations in time, (1) is a natural energy density to consider.

From the Maxwell equation

$$\frac{\partial E_k}{\partial t} = 4\pi e\int f_k v\,dv \qquad 6$$

it readily follows that

$$E_{-k}\frac{\partial}{\partial t}E_k + (k\to -k) = 4\pi e\int v\left(E_{-k}f_k + (k\to -k)\right)dv. \qquad 7$$

Using (2) to eliminate E_k on the right side of (7) this is readily rewritten

$$\frac{\partial}{\partial t}|\hat{E}_k|^2 = 4\pi m\int v\,\frac{1}{\partial f_o/\partial v}\,\frac{\partial}{\partial t}|f_k|^2 dv\,. \qquad 8$$

We thus define from (8) an energy density

$$\frac{|\hat{E}_k|^2}{8\pi} - \frac{4\pi m}{8\pi}\int v\,\frac{|f_k|^2 dv}{\partial f_o/\partial v}\,. \qquad 9$$

From (3) this becomes

$$\frac{|\hat{E}_k|^2}{8\pi}\left(1 - \frac{4\pi e^2}{m k}\int \frac{k v\,\partial f/\partial v}{(\omega_k - kv)^2}\,dv\right)$$

$$= \frac{|\hat{E}_k|^2}{8\pi}\left(1 + \frac{4\pi e^2}{mk}\int \frac{\partial f/\partial v}{(\omega_k - kv)}\,dv - \omega_k\frac{4\pi e^2}{mk}\int\frac{\partial f/\partial v}{(\omega_k - kv)^2}\,dv\right).$$

Using (5), this is just

$$= \frac{|\hat{E}_k|^2}{8\pi}\left(-\omega_k\frac{4\pi e^2}{mk}\int\frac{\partial f/\partial v}{(\omega_k - kv)^2}\,dv\right) \qquad 10$$

$$= \frac{|\hat{E}_k|^2}{8\pi}\left(\omega_k\left.\frac{\partial \mathcal{E}_k}{\partial \omega}\right|_{\omega_k}\right).$$

This represents the field plus polarization energy. (8) is just a statement that this energy isconserved. If we had included the effect of a small instability there would be additional driving terms on the right side of (8), however the natural energy density occuring in the analysis would still be that given in (10).

Since the energy $\frac{|\hat{E}_k|^2}{8\pi} \omega_k \left.\frac{\partial \varepsilon_k}{\partial \omega}\right|_{\omega_k}$ is partly made up of field energy and partly of polarization energy due to the interaction of the waves and particles it is convenient to introduce the concept of "quasi-particles" (although "quasi-waves" may be a more appropiate nomenclature). The number of quasi-particles in the k-th mode is defined to be n_k where

$$n_k \omega_k = \frac{|\hat{E}_k|^2}{8\pi} \omega_k \left.\frac{\partial \varepsilon_k}{\partial \omega}\right|_{\omega_k},$$

i.e. n_k is the net wave energy in the k-th mode divided by the frequency of the k-th mode.

APPENDIX H_1

The Method of Galeev and Karpman[16]

In considering the Eq. (V-1)

$$\frac{\partial}{\partial t} E_{\underline{k}} + i\omega_{\underline{k}} E_{\underline{k}} = \int d\underline{k}' \, K(\underline{k}', \underline{k}-\underline{k}') \, E_{\underline{k}'} E_{\underline{k}-\underline{k}'} \, , \qquad 1$$

the authors treat the nonlinear terms as small and obtain a perturbation solution of the form

$$\hat{E}_{\underline{k}} = \lambda \hat{E}_{\underline{k}}^{(1)} + \lambda^2 \hat{E}_{\underline{k}}^{(2)} + \lambda^3 \hat{E}_{\underline{k}}^{(3)} + \cdots \qquad 2$$

where $E_k = \hat{E}_k e^{-i\omega_k t}$, and $\omega_{\underline{k}}$ is independent of t. Because of the structure of the original equation they obtain $\hat{E}_k^{(2)}$ as an integral functional of the product of two $\hat{E}_k^{(1)}$'s and $\hat{E}_k^{(3)}$ as an integral functional of the product of three $\hat{E}_k^{(1)}$'s and so on. To obtain a kinetic equation for the waves they then average the difference

$$\hat{E}_k \hat{E}_{k'} - \hat{E}_k^{(1)} \hat{E}_{k'}^{(1)} \qquad \text{over} \qquad 3$$

an ensemble in which the phases of the $\hat{E}_k$ for different k are assumed random, keeping terms to order λ^4. Their prescription for averaging (where < > denotes an average over the ensemble) is the following :

$$\langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(1)} \rangle = \mathcal{E}_{\underline{k}}^{(1)} \delta(\underline{k} + \underline{k}') \, , \qquad 4$$

$$\langle \hat{E}_{\underline{k}}^{(1)} \hat{E}_{\underline{k}'}^{(1)} \hat{E}_{\underline{k}''}^{(1)} \rangle = 0 \, , \qquad 5$$

and

$$\langle \hat{E}^{(1)}_{\underline{k}} \hat{E}^{(1)}_{\underline{k}'} \hat{E}^{(1)}_{\underline{k}''} \hat{E}^{(1)}_{\underline{k}'''} \rangle$$

$$= \mathcal{E}^{(1)}_{\underline{k}} \mathcal{E}^{(1)}_{\underline{k}''} \delta(\underline{k}+\underline{k}')\delta(\underline{k}''+\underline{k}''')$$

$$+ \mathcal{E}^{(1)}_{\underline{k}} \mathcal{E}^{(1)}_{\underline{k}''} \delta(\underline{k}+\underline{k}''')\delta(\underline{k}'+\underline{k}'') + \mathcal{E}^{(1)}_{\underline{k}} \mathcal{E}^{(1)}_{\underline{k}'} \delta(\underline{k}+\underline{k}'')\delta(\underline{k}'+\underline{k}''') . \tag{6}$$

As discussed in Chapter V, this is manifestly compatible with the homogeneous nature of the ensemble. Effectively they are viewing the product of 3 $E_k^{(1)}$'s as the product of three random numbers which they take as zero (see also Ref. 10), and the product of 4 $E_k^{(1)}$'s by taking the average in all possible pairs. This method of averaging is quite prevalent in the literature(10), (7), (16) not only in the solution of the equation discussed in this section but also in the treatment of homogeneous turbulence in an ensemble of Vlasov plasmas. We briefly discuss this technique of averaging in Appendix H_4 and do not comment on it further at this point.

Averaging 3 according to the prescription 4 - 6 the authors find that $\mathcal{E}_k - \mathcal{E}_k^{(1)}$ varies linearly in t . From this they construct the change in $\mathcal{E}_k$ per unit time $\dfrac{\mathcal{E}_k - \mathcal{E}_k^{(1)}}{t}$, (denote $(\dfrac{\partial \mathcal{E}_k}{\partial t})$) i.e., the transition probability per unit time. The resulting equation is similar to Eq. 32 of Chapter V, and also that obtained in Ref. 15 , describing the change of $\mathcal{E}_k$ due to wave-wave interactions.

APPENDIX H_2

"Simple" and "Multiple" Resonances.

We preface our comments on the approach of Benney and Saffman[15] with a discussion of the assumptions inherent in the derivation of Chap. V, namely

1. For large t, integrals of the form

$$\int d\underline{k}' f(\underline{k},\underline{k}') \frac{e^{i(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'})t} - 1}{i(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'})}, \qquad 1$$

where f is a relatively smooth function, behave effectively as if

$$\frac{e^{i(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'})t} - 1}{i(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'})} \longrightarrow \frac{i}{\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'} + i\delta}, \quad \delta \to 0_+ . \qquad 2$$

2. Also, in order that integrals over k', involving $\delta(\omega_k - \omega_{k'} - \omega_{k-k'})$ would not diverge it was assumed that

$$\frac{d}{dk'}(\omega_{k'} + \omega_{k-k'}) = 0, \qquad 3$$

and

$$\omega_{k'} + \omega_{k-k'} = \omega_k \quad , \qquad 4$$

could not be satisfied simultaneously for some k' (given k). That is, to say we assume the resonance as "simple" as opposed to "multiple".

The condition that Eqs. 3 and 4 cannot be satisfied simultaneously is in fact a necessary condition that 2 be valid.

We consider for simplicity

$$U(t) = \int g(x) \frac{e^{if(x)t} - 1}{i f(x)} dx \,, \tag{5}$$

where f is a real function of the real variable x and g may be complex. It is assumed that the x integration spans the entire range and that the functions are sufficiently well behaved that the integral exists.

We first examine

$$\frac{\partial U}{\partial t}(t) = \int g(x)\, e^{if(x)t} dx \,, \tag{6}$$

and assume temporarily that

$$f'(x) = 0 \,, \tag{7}$$

and

$$f(x) = 0 \,, \tag{8}$$

can be satisfied simultaneously for some x , say x_o . The problem of looking at 6 for large t then lends itself readily to the method of stationary phase, i.e., we expect the major contribution to the integral to be coming from the region where the phase is changing most slowly, namely x_o . We thus Taylor expand f(x) about x_o. Assuming $f''(x_o) \neq 0$ the leading term is

$$\frac{1}{2} f''(x_o)(x - x_o)^2 + \ldots \quad .$$

Then

$$\frac{\partial U}{\partial t} \simeq \int g(x)\, e^{i \frac{1}{2} f''(x_o)(x-x_o)^2 t} dx \tag{9}$$

Changing variables (and for convenience taking $f''(x_o) > 0$) to

$$y = \sqrt{\frac{f''(x_0)t}{2}}\,(x-x_0)\,,$$

$$\frac{\partial U}{\partial t} \simeq \sqrt{\frac{2}{f''(x_0)t}}\int dy\, g\left(x_0 + \sqrt{\frac{2}{f''(x_0)t}}\,y\right) e^{iy^2} \tag{10}$$

$$\simeq \sqrt{\frac{2\pi}{f''(x_0)t}}\, g(x_0)\, e^{i\pi/4}\,, \qquad \text{for large } t \tag{11}$$

and hence apart from a constant, for large t

$$U(t) \sim 2\sqrt{\frac{2\pi}{f''(x_0)}}\, t^{1/2}\, e^{i\pi/4} g(x_0)\,, \tag{12}$$

which diverges. The situation is quite different if Eqs. 7 and 8 cannot be satisfied simultaneously. For example if $f'(x_o) = 0$

but $f(x_o) \neq 0$,

then for large t, Eq. 11 becomes

$$\frac{\partial}{\partial t} U(t) \sim \sqrt{\frac{2\pi}{f''(x_0)t}}\, e^{if(x_0)t} g(x_0)\, e^{i\pi/4}. \tag{13}$$

Apart from an additive constant, for large t, 13 gives to leading order

$$U(t) \sim \frac{1}{if(x_0)}\sqrt{\frac{2\pi}{f''(x_0)t}}\, g(x_0)\, e^{if(x_0)t}\, e^{i\pi/4}\,, \tag{14}$$

which decays effectively as $1/t^{1/2}$, i.e.,

$$U(t) \simeq U_\infty + O(1/t^{1/2}). \tag{15}$$

To avoid the divergent behavior which occurs in Eq. 12 for large t, we thus assume that

$f'(x) = 0$ and $f(x) = 0$ cannot be satisfied simultaneously,

or in the problem at hand that $\frac{d}{dk'}(\omega_{k'} + \omega_{k-k'}) = 0$

and $\omega_{k'} + \omega_{k-k'} = \omega_k$ cannot be

satisfied simultaneously for some k' (given k). With this assumption it is relatively simple to show that for large t, $\frac{e^{if(x)t} - 1}{i f(x)}$ in the integrand of Eq. 5 behaves effectively as $\frac{i}{f(x) + i\delta}$ or $\pi\delta(f(x)) + i \frac{P}{f(x)}$. The $\sin \frac{f(x)t}{f(x)}$ peaks for large t about the solutions of $f(x) = 0$ and behaves essentially as $\pi\delta(f(x))$ as far as the integral is concerned. The $\frac{\cos f(x)t - 1}{i\, f(x)}$ behaves like $i \frac{P}{f(x)}$ for large t. The $\cos f(x)t$ oscillates rapidly except about the solutions of $f(x) = 0$. But for those x, $\cos f(x)t - 1 = 0$. Thus those x for which $f(x) = 0$ are omitted from the integration and $\frac{\cos f(x)t - 1}{i\, f(x)}$ behaves effectively as $i \frac{P}{f(x)}$.

Then for large t

$$U(t) \sim \int dx\; g(x) \frac{i}{f(x) + i\delta} + \text{corrections which are functions of } t. \tag{16}$$

In the case that $f(x)$ has a point of stationary phase (an x such that $f'(x) = 0$) but that $f(x) \neq 0$ for that point, the leading corrections are of the form 14 i.e.,

$$\frac{\text{(oscillation)}}{t^{1/2}} . \tag{17}$$

In the case that $f(x)$ has <u>no</u> point of stationary phase we expect that the leading correction may decay even more rapidly. For example in looking at

$$\int g(x)\, e^{if(x)t} dx ,$$

if we transform x such that $y = f(x)$ (assume that this is 1 to 1 for simplicity) then the integral is effectively

$$\frac{\partial U(t)}{\partial t} \simeq \int G(y)\, e^{iyt} dy \tag{18}$$

where

$$G(y) = \frac{g(x(y))}{\frac{df}{dx}(y)} .$$

The range of integration depends on $f(x)$ as $x \longrightarrow \pm\infty$. We recognize that Eq. 18 is just the Fourier transform of $G(y)$ over some range. Depending on the properties of $G(y)$ and the range of integration this may decay faster than in Eq. 13 , i.e., than

$$\frac{\text{(oscillation)}}{t^{1/2}} .$$

For instance, if for large t the transform in 18 varies as $e^{-\beta t}$, ,
then the corrections in Eq. 16 are of the form $e^{-\beta t}$. In any case, barring some pathological behavior of the functions involved, we expect that for large t, 2 is a good approximation in the integral 1, provided 3 and 4 cannot be satisfied simultaneously.

APPENDIX H_3

The Method of Benney and Saffman[15] Revised.

In solving the equation

$$\frac{\partial}{\partial t}\hat{E}_{\underline{k}} = \int d\underline{k}'\, K(\underline{k}', \underline{k}-\underline{k}')\, \hat{E}_{\underline{k}'}\, \hat{E}_{\underline{k}-\underline{k}'}\, e^{i(\omega_{\underline{k}}-\omega_{\underline{k}'}-\omega_{\underline{k}-\underline{k}'})t}, \qquad 1$$

the authors use a multiple-time perturbation expansion

$$\hat{E}_{k} \simeq \lambda\, \hat{E}_{\underline{k}}^{(1)}(t, \lambda t \cdots) + \lambda^2\, \hat{E}_{\underline{k}}^{(2)}(t, \lambda t \cdots) + \cdots \qquad 2$$

As they are interested in certain averages characteristic of the spatially homogeneous ensemble, secularities are not removed in the $\hat{E}_k^{(2)}$ and $\hat{E}_k^{(3)}$ in the usual manner. Rather, all terms are retained in these quantities but in asking that certain ensemble averages be given in terms of expressions which are uniformly valid, the freedom characteristic of the multiple time technique is then used to remove secular behavior and determine the evolution of ensemble quantities on longer time scales.

We agree with their basic philosophy but differ in opinion on certain points. In particular their conclusions concerning behavior on the λt time scale are inadequate. This is a reflection of the fact that they demand that the expression for

$$\left\langle \hat{E}_k^{(2)}\ \hat{E}_{k'}^{(2)} \right\rangle$$

be uniformly valid per se, rather than the total relevant quantity associated with the λ^4 order, namely,

$$\left[\left\langle \hat{E}_k^{(1)}\, \hat{E}_{k'}^{(3)} \right\rangle + (\underline{k} \Leftrightarrow k') \right] + \left\langle \hat{E}_k^{(2)}\, \hat{E}_{k'}^{(2)} \right\rangle .$$

In addition they take the initial value of $\hat{E}_k^{(2)}$ on the short-time scale (viz. $\hat{E}_k^{(2)}(0, \lambda t \ldots)$) as being zero which is not a priori justified in the many-time theory.

One of the main features of the derivation presented in Chapter V (where ensemble quantities are advanced at the outset and then solutions obtained) was the fact that the natural time scales occurring in the analysis were t, $\lambda^2 t$, not t, λt, $\lambda^2 t$. It is thus pertinent to present a rigorous analysis of the solution to 1 which will clarify the role of the λt scale as far as ensemble quantities are concerned.

The relevant equations are

$$\frac{\partial}{\partial t} \hat{E}_{\underline{k}}^{(1)} = 0 \qquad 3$$

$$\frac{\partial}{\partial t} \hat{E}_{\underline{k}}^{(2)} + \frac{\partial}{\partial \lambda t} \hat{E}_{\underline{k}}^{(1)} = \int d\underline{k}' \, K(\underline{k}', \underline{k}-\underline{k}') \hat{E}_{\underline{k}'} \hat{E}_{\underline{k}-\underline{k}'} e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} \qquad 4$$

$$\frac{\partial}{\partial t} \hat{E}_k^{(3)} + \frac{\partial}{\partial \lambda t} \hat{E}_k^{(2)} + \frac{\partial}{\partial \lambda^2 t} \hat{E}_k^{(1)}$$

$$= 2 \int d\underline{k}' \, K(\underline{k}', \underline{k}-\underline{k}') \hat{E}_{\underline{k}'}^{(1)} \hat{E}_{\underline{k}-\underline{k}'}^{(2)} e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} \qquad 5$$

$$\vdots$$

From Eq. 3 with $\hat{E}_k^{(1)} = \hat{E}_k^{(1)}(\lambda t \ldots)$, Eq. 4 yields

$$\hat{E}_{k}^{(2)}(t,\lambda t\cdots) = \hat{E}_{\underline{k}}^{(2)}(0,\lambda t\cdots) - t\frac{\partial}{\partial\lambda t}\hat{E}_{\underline{k}}^{(1)}(0,\lambda t\cdots)$$

$$+\int d\underline{k}'\, K(\underline{k}',\underline{k}-\underline{k}')\hat{E}_{\underline{k}-\underline{k}'}^{(1)}\hat{E}_{\underline{k}'}^{(1)}\int_0^t e^{i(\omega_k-\omega_{k'}-\omega_{k-k'})t}\,dt\,. \qquad 6$$

The expression for $\hat{E}_k^{(2)}$ in Eq. 6 when substituted into Eq. 5, gives for $\hat{E}_k^{(3)}$

$$\hat{E}_{\underline{k}}^{(3)}(t,\lambda t\cdots) = \hat{E}_{\underline{k}}^{(3)}(0,\lambda t\cdots) - t\frac{\partial}{\partial\lambda t}\hat{E}_{\underline{k}}^{(2)}(0,\lambda t\cdots)$$

$$+\frac{t^2}{2}\frac{\partial^2}{\partial\lambda t^2}\hat{E}_{\underline{k}}^{(1)}(0,\lambda t\cdots) - t\frac{\partial}{\partial\lambda^2 t}\hat{E}_{\underline{k}}^{(1)}(0,\lambda t\cdots)$$

$$-t\frac{\partial}{\partial\lambda t}\int d\underline{k}'\, K(k',\underline{k}-\underline{k}')\hat{E}_{\underline{k}-\underline{k}'}^{(1)}\hat{E}_{\underline{k}'}^{(1)}\left(\int_0^t e^{i(\omega_k-\omega_{k'}-\omega_{k-k'})t}\,dt\right)$$

$$+2\int d\underline{k}'\, K(\underline{k}',\underline{k}-\underline{k}')\hat{E}_{\underline{k}-\underline{k}'}^{(1)}\hat{E}_{\underline{k}'}^{(2)}(0,\lambda t)\left(\int_0^t e^{i(\omega_k-\omega_{k'}-\omega_{k-k'})t}\,dt\right)$$

$$+2\iint d\underline{k}'\,d\underline{k}''\, K(\underline{k}',\underline{k}-\underline{k}')\,K(\underline{k}'',\underline{k}'-\underline{k}'')\hat{E}_{k-k'}^{(1)}\hat{E}_{k'-k''}^{(1)}\hat{E}_{k''}^{(1)}$$

$$\times\left(\int_0^t e^{i(\omega_k-\omega_{k'}-\omega_{k-k'})t}\,dt \times \int_0^t e^{i(\omega_{k'}-\omega_{k''}-\omega_{k'-k''})t'}\,dt'\right). \qquad 7$$

We have interchanged orders of integration quite freely above. In contrast to Ref. 15 we allow the initial values of $\hat{E}_k^{(2)}$ and $\hat{E}_k^{(3)}$ on the short-time scale to depend on $\lambda t\ldots$ (in lieu of taking the initial values as zero).

A meaningful quantity to be averaged over the ensemble is $E_k E_{k'}$, that is

$$\left\langle \left(\lambda \hat{E}_k^{(1)} + \lambda^2 \hat{E}_k^{(2)} + \lambda^3 \hat{E}_k^{(3)} + \cdots \right)\left(\lambda \hat{E}_{k'}^{(1)} + \lambda^2 \hat{E}_{k'}^{(2)} + \lambda^3 \hat{E}_{k'}^{(3)} + \cdots \right)\right\rangle \qquad 8$$

To order λ^4 this is just

$$\begin{aligned} &\lambda^2 \left\{ \langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(1)} \rangle \right\} \\ &+ \lambda^3 \left\{ \langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(2)} \rangle + (k \leftrightarrow k') \right\} \\ &+ \lambda^4 \left\{ \left[\langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(3)} \rangle + (k \leftrightarrow k') \right] + \langle \hat{E}_k^{(2)} \hat{E}_{k'}^{(2)} \rangle \right\} \\ &+ \quad \vdots \end{aligned} \qquad 9$$

To ensure the ordering remains intact, we ask that each of the following quantities be given by uniformly valid expressions

$$\langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(2)} \rangle + (k \leftrightarrow k') \qquad 10$$

and

$$\left\{ \langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(3)} \rangle + (k \leftrightarrow k') \right\} + \langle \hat{E}_k^{(2)} \hat{E}_{k'}^{(2)} \rangle . \qquad 11$$

The freedom inherent in the many-time analysis will allow us to remove secular behavior and ensure that this is true. The method of averaging over the ensemble is the prescription given in V-15, -16, and -17.

We introduce the notation

$$\langle \hat{E}_{\underline{k}}^{(1)} \hat{E}_{\underline{k}'}^{(1)} \rangle = \mathcal{E}_{\underline{k}}(\lambda t \cdots)\, \delta(\underline{k}+\underline{k}')$$

$$\langle \hat{E}_{\underline{k}}^{(1)} \hat{E}_{\underline{k}'}^{(1)} \hat{E}_{\underline{k}''}^{(1)} \rangle = T_{k,k'}(\lambda t \cdots)\, \delta(\underline{k}+\underline{k}'+\underline{k}'')$$

$$\langle \hat{E}_{\underline{k}}^{(1)} \hat{E}_{\underline{k}'}^{(2)}(0,\lambda t,\cdots) \rangle = \mathcal{J}_{k,k'}^{(1),(2)}(\lambda t,\cdots)\, \delta(\underline{k}+\underline{k}')$$

$$\langle \hat{E}_{\underline{k}}^{(1)} \hat{E}_{\underline{k}'}^{(3)}(0,\lambda t,\cdots) \rangle = \mathcal{J}_{k,k'}^{(1),(3)}(\lambda t,\cdots)\, \delta(\underline{k}+\underline{k}')$$

$$\begin{aligned}\langle \hat{E}_{k}^{(1)} \hat{E}_{k'}^{(1)} \hat{E}_{k''}^{(1)} \hat{E}_{k'''}^{(1)} \rangle &= F_{k,k',k''}(\lambda t,\cdots)\, \delta(\underline{k}+\underline{k}'+\underline{k}''+\underline{k}''') \\ &+ \mathcal{E}_{\underline{k}}(\lambda t,\cdots)\, \mathcal{E}_{\underline{k}''}(\lambda t,\cdots)\, \delta(\underline{k}+\underline{k}')\, \delta(\underline{k}''+\underline{k}''') \\ &+ \mathcal{E}_{\underline{k}}(\lambda t,\cdots)\, \mathcal{E}_{\underline{k}'}(\lambda t,\cdots)\, \delta(\underline{k}+\underline{k}'')\, \delta(\underline{k}'+\underline{k}''') \\ &+ \mathcal{E}_{\underline{k}}(\lambda t,\cdots)\, \mathcal{E}_{\underline{k}'}(\lambda t,\cdots)\, \delta(\underline{k}+\underline{k}''')\, \delta(\underline{k}'+\underline{k}'')\end{aligned} \qquad 12$$

$$\langle \hat{E}_{\underline{k}_1}^{(2)}(0,\lambda t,\cdots) \hat{E}_{\underline{k}_2}^{(1)} \hat{E}_{\underline{k}_3}^{(1)} \rangle = T_{\underline{k}_1,\underline{k}_2}^{(2),(1),(1)}(\lambda t,\cdots)\, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3)$$

$$\langle \hat{E}_{\underline{k}}^{(2)}(0,\lambda t,\cdots) \hat{E}_{\underline{k}'}^{(2)}(0,\lambda t,\cdots) \rangle = \mathcal{J}_{\underline{k},\underline{k}'}^{(2),(2)}(\lambda t,\cdots)\, \delta(\underline{k}+\underline{k}') .$$

Equation 6, when substituted into expression 11, yields

$$\left\langle \hat{E}^{(2)}_{\underline{k}}(t,\lambda t\cdots)\, \hat{E}^{(1)}_{\underline{k}'} \right\rangle + (\underline{k} \leftrightarrow \underline{k}')$$

$$= \left\langle \hat{E}^{(2)}_{\underline{k}}(0,\lambda t\cdots)\, \hat{E}^{(1)}_{\underline{k}'} \right\rangle + (\underline{k} \leftrightarrow \underline{k}')$$

$$- t \left\langle \hat{E}^{(1)}_{\underline{k}'} \frac{\partial}{\partial \lambda t} \hat{E}^{(1)}_{\underline{k}} \right\rangle + (\underline{k} \leftrightarrow \underline{k}')$$

$$+ \left\{ \int d\underline{k}''\, K(\underline{k}'', \underline{k}-\underline{k}'') \left\langle \hat{E}^{(1)}_{k'}\, \hat{E}^{(1)}_{k-k''}\, \hat{E}^{(1)}_{k''} \right\rangle \frac{e^{i(\omega_k - \omega_{k''} - \omega_{k-k''})t} - 1}{i(\omega_k - \omega_{k''} - \omega_{k-k''})} \right.$$

$$\left. + (\underline{k} \leftrightarrow \underline{k}') \right\} \qquad 13$$

$$= \left\{ \delta(\underline{k}+\underline{k}') \left[\mathcal{J}^{(2),(1)}_{\underline{k},\underline{k}'}(\lambda t, \cdots) + \int d\underline{k}''\, K(\underline{k}'', \underline{k}-\underline{k}'')\, T_{k',k-k''}(\lambda t) \int_0^t e^{i(\omega_k - \omega_{k''} - \omega_{k-k''})t} dt \right. \right.$$

$$\left. \left. + (\underline{k} \leftrightarrow \underline{k}') \right] \right\}$$

$$- t\, \delta(\underline{k}+\underline{k}') \frac{\partial}{\partial \lambda t} \mathcal{E}_{\underline{k}} \,, \qquad 14$$

Under the conditions discussed in Appendix H_2, for large t we replace

$$\frac{e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} - 1}{i(\omega_k - \omega_{k'} - \omega_{k-k'})} \quad \text{by} \quad \frac{i}{\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta}$$

in integrals over k' throughout the remainder of this section, i.e., for large t, 14 becomes

$$\left\{ \delta(\underline{k}+\underline{k}') \left(\vartheta^{(2)}_{k,k'}(\lambda t,\cdots) + \int d\underline{k}'' \, K(k'',k-k'') \frac{T_{k',k-k''}(\lambda t\cdots)\ (i)}{\omega_k - \omega_{k''} - \omega_{k-k''} + i\delta} \right) \right.$$

$$\left. + \ (\underline{k} \leftrightarrow \underline{k}') \right\}$$

$$- t \ \delta(\underline{k}+\underline{k}_1) \frac{\partial}{\partial \lambda t} \mathcal{E}_k \ . \qquad 15$$

Our uniformity criterion leads us to conclude that

$$\frac{\partial}{\partial \lambda t} \mathcal{E}_k = 0 \ , \qquad 16$$

provided

$$\left\{ \vartheta^{(2),(1)}_{k,-k}(\lambda t) + \int d\underline{k}'' \, K(\underline{k}'',\underline{k}-\underline{k}'') T_{-k,k-k''}(\lambda t\cdots) \frac{i}{\omega_k - \omega_{k''} - \omega_{k-k''} + i\delta} \right\}$$

$$+ \ (\underline{k} \rightarrow -\underline{k}) \qquad 17$$

is not ill-behaved for large λt. In the next order we will in fact show, assuming 16, that $\frac{\partial}{\partial \lambda t}$ of the expression in 17 is zero. 16 tells us that there is no systematic change in the lowest-order energy density on the time scale, λt. In calculating 11, using Eqs. 7, 6, and 12, together with the properties of K given in Chapter V (V-9), one finds after considerable algebraic manipulation that

$$\left\{ \langle \hat{E}_{\underline{k}}^{(1)} \hat{E}_{\underline{k}_3}^{(1)} \rangle + (\underline{k} \leftrightarrow \underline{k}_3) \right\} + \langle \hat{E}_{\underline{k}}^{(2)} \hat{E}_{\underline{k}_3}^{(2)} \rangle$$

$$= \frac{t^2}{2} \delta(\underline{k}+\underline{k}_3) \frac{\partial}{\partial \lambda t}\left(\frac{\partial}{\partial \lambda t} \mathcal{E}_{\underline{k}} \right)$$

$$+ t\, \delta(\underline{k}+\underline{k}_3) \Bigg\{ - \frac{\partial}{\partial \lambda t} \left(\vartheta^{(2),(1)}_{\underline{k},\underline{k}_3} (\lambda t \cdots) + i \int d\underline{k}' \frac{K(\underline{k}', \underline{k}-\underline{k}') T_{\underline{k}_3, \underline{k}-\underline{k}'}(\lambda t \cdots)}{\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta} \right)$$

$$- \frac{\partial}{\partial \lambda t} (\underline{k} \leftrightarrow \underline{k}_3) - \frac{\partial}{\partial \lambda^2 t} \mathcal{E}_{\underline{k}}$$

$$+ 4\pi \int d\underline{k}' \, \mathcal{E}_{\underline{k}'} \mathcal{E}_{\underline{k}-\underline{k}'} \, | K(\underline{k}', \underline{k}-\underline{k}') |^2 \, \delta(\omega_k - \omega_{k'} - \omega_{k-k'})$$

$$- 8 \mathcal{I}m \int d\underline{k}' \, \mathcal{E}_{\underline{k}} \mathcal{E}_{\underline{k}-\underline{k}'} \frac{K(\underline{k}', \underline{k}-\underline{k}') K(\underline{k}, \underline{k}'-\underline{k})}{\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'} + i\delta} \Bigg\} \qquad 18$$

$$+ \delta(\underline{k}+\underline{k}_3) \Bigg\{ \left(\vartheta^{(1),(3)}_{\underline{k}_3,\underline{k}} (\lambda t \cdots) + (\underline{k} \leftrightarrow \underline{k}_3) \right) + \left(\vartheta^{(2),(2)}_{\underline{k}_3,\underline{k}} (\lambda t, \cdots) \right)$$

$$+ 2 \left(\int d\underline{k}' \frac{K(\underline{k}', \underline{k}-\underline{k}') T^{(2),(1),(1)}_{\underline{k}', \underline{k}-\underline{k}'} (\lambda t \cdots)}{\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta} (i) + (\underline{k} \leftrightarrow \underline{k}_3) \right)$$

$$+ \left(\int d\underline{k}' \frac{K(\underline{k}', \underline{k}-\underline{k}') T^{(2),(1),(1)}_{\underline{k}_3, \underline{k}-\underline{k}'}}{\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta} (i) + (\underline{k} \leftrightarrow \underline{k}_3) \right)$$

$$+ 2 \left(- \iint d\underline{k}' d\underline{k}'' \frac{K(\underline{k}', \underline{k}-\underline{k}') K(k'', k'-k'') F_{k_3, k-k', k'-k''}(\lambda t \cdots)}{(\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta)(\omega_{k'} - \omega_{k''} - \omega_{k'-k''} + i\delta)} + (\underline{k} \leftrightarrow \underline{k}_3) \right)$$

$$+ \left(- \iint d\underline{k}' d\underline{k}'' \frac{K(\underline{k}', \underline{k}-\underline{k}') K(k'', \underline{k}_3-\underline{k}'') F_{\underline{k}-\underline{k}', \underline{k}', k_3-k''}(\lambda t \cdots)}{(\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta)(\omega_{k_3} - \omega_{k''} - \omega_{k_3-k''} + i\delta)} \right) \Bigg\}$$

In deriving a kinetic equation for the waves the last term in 18 is of little interest (the term not involving t or t^2). We only ask that it be well-behaved and the functions be such that the different k integrals exist. In fact if we were to assume (as in Ref. 15)

$$\hat{E}_k^{(2)}(0, \lambda t \ldots) = 0 ,$$
$$\hat{E}_k^{(3)}(0, \lambda t \ldots) = 0 ,$$

and that the irreducible 4-correlations (F) were effectively zero (as in Ref. 16), the entire term would vanish. Our uniformity criterion demands that the t^2 term vanish giving us

$$\frac{\partial}{\partial \lambda t}\left(\frac{\partial}{\partial \lambda t} \mathcal{E}_k\right) = 0 , \qquad 19$$

which reaffirms Eq. 16 . As regards the t term in Eq. 18 we imagine integrating over k_3 . Removing the t secularity gives us that

$$\left\{ \frac{\partial}{\partial \lambda t}\left(\mathcal{V}^{(a),(1)}_{\underline{k},-\underline{k}}(\lambda t, \cdots) + i \int d\underline{k}' \frac{K(\underline{k}', \underline{k}-\underline{k}')\, T_{-\underline{k}, \underline{k}-\underline{k}'}(\lambda t, \cdots)}{(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'} + i\delta)} \right)\right\}$$

$$+ \left\{ \underline{k} \rightarrow -\underline{k} \right\}$$

$$+ \frac{\partial}{\partial \lambda^2 t} \mathcal{E}_{\underline{k}} - 4\pi \int d\underline{k}' \, \mathcal{E}_{\underline{k}'} \mathcal{E}_{\underline{k}-\underline{k}'} \left| K(\underline{k}', \underline{k}-\underline{k}')\right|^2 \delta(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'})$$

$$+ 8\, \mathcal{I}m \int d\underline{k}' \, \mathcal{E}_{\underline{k}} \mathcal{E}_{\underline{k}-\underline{k}'} \frac{K(\underline{k}', \underline{k}-\underline{k}')\, K(\underline{k}, \underline{k}'-\underline{k})}{(\omega_k - \omega_{k'} - \omega_{k-k'} + i\delta)} \qquad 20$$

$$\equiv 0 .$$

If we were to integrate this with respect to λt, then since $\mathcal{E}_k$ is independent of λt, the last two terms of the above expression would lead to terms proportional to λt and give secular behavior for large λt unless

$$\frac{\partial}{\partial \lambda^2 t}\mathcal{E}_{\underline{k}} = 4\pi \int d\underline{k}'\, \mathcal{E}_{\underline{k}'}\mathcal{E}_{\underline{k}-\underline{k}'}\,|K(\underline{k}',\underline{k}-\underline{k}')|^2\,\delta(\omega_{\underline{k}}-\omega_{\underline{k}'}-\omega_{\underline{k}-\underline{k}'})$$

$$- 8\,\mathcal{I}m\int d\underline{k}'\,\mathcal{E}_{\underline{k}}\mathcal{E}_{\underline{k}-\underline{k}'}\,\frac{K(\underline{k}',\underline{k}-\underline{k}')\,K(\underline{k},\underline{k}'-\underline{k})}{(\omega_{k}-\omega_{k'}-\omega_{k-k'}+i\delta)}, \qquad 21$$

which is just the kinetic equation giving us the change of lowest-order energy density $\mathcal{E}_k$, on the $\lambda^2 t$ time scale. This is identical with the result given in V-32. We also have the additional information that

$$\frac{\partial}{\partial\lambda t}\left(\mathcal{J}^{(2),(1)}_{\underline{k},-\underline{k}}(\lambda t\cdots) + i\int d\underline{k}'\,\frac{K(\underline{k}',\underline{k}-\underline{k}')\,T_{-\underline{k},\underline{k}-\underline{k}'}(\lambda t\cdots)}{(\omega_k-\omega_{k'}-\omega_{k-k'}+i\delta)}\right)$$

$$+(\underline{k}\to-\underline{k}) = 0\,. \qquad 22$$

<u>Discussion of results:</u>

The quantity of physical interest $\mathcal{E}_k$, which is essentially the energy density associated with the k'th mode, does not vary on the λt scale. Its kinetic behavior is given by Eq. 21 which describes its evolution on the $\lambda^2 t$ scale. The information in Eq. 22 on the λt scale seems rather extraneous and of little physical interest. In fact if we take, as in Ref. 15

$$E_k^{(2)}(0,\ \lambda t\ldots) = 0\ ,$$

then Eq. 22 yields

$$\frac{\partial}{\partial \lambda t} \int d\underline{k}' \; \frac{K(\underline{k}', \underline{k}-\underline{k}')\, T_{-\underline{k}, \underline{k}-\underline{k}'}(\lambda t \cdots)}{(\omega_{\underline{k}} - \omega_{\underline{k}'} - \omega_{\underline{k}-\underline{k}'} + i\delta)} \qquad + (\underline{k} \to -\underline{k})$$

$$= 0 \tag{23}$$

A sufficient condition to ensure that this is true (see Eq. 12) is

$$\frac{\partial}{\partial \lambda t} \langle \hat{E}^{(1)}_{\underline{k}_1} \hat{E}^{(1)}_{\underline{k}_2} \hat{E}^{(1)}_{\underline{k}_3} \rangle = 0$$

i.e., $\langle \hat{E}^{(1)}_{k1} \; \hat{E}^{(1)}_{k2} \; \hat{E}^{(1)}_{k3} \rangle$ does not vary on the t of λt scale. Any time variations of this quantity would then appear in effects of higher order than λ^4 which is beyond the scope of our present calculations.

It is of considerable import to note the general validity of the final kinetic Eq. 21 . It is true whether or not we assume

$$E^{(2)}_k (0, \lambda t \ldots) = 0 \qquad \text{(as in Ref. 15)} \tag{24}$$

and/or $\left\langle \hat{E}^{(1)}_{k1} \; \hat{E}^{(1)}_{k2} \; \hat{E}^{(1)}_{k3} \right\rangle = 0$ (as in Refs. 16 and 10) 25

However, if we were to accept 24 (by assuming that the initial value of $E^{(2)}_k$ is independent of λ) then to the order relevant in this theory (λ^4) 25 would remain true if true initially.

APPENDIX H_4

Ensemble Averaging Prevalent in the Russian Literature

In problems of weak turbulence in a Vlasov plasma or in systems satisfying dynamical equations of the form Eq. 1, Chapter V, the prescription for averaging over a statistical ensemble has often been[7] [10] [16]

$$\langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(1)} \rangle = \mathcal{E}_k \delta(\underline{k}+\underline{k}') , \tag{a}$$

$$\langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(1)} \hat{E}_{k''}^{(1)} \rangle = 0 , \tag{b}$$

$$\begin{aligned} \langle \hat{E}_k^{(1)} \hat{E}_{k'}^{(1)} \hat{E}_{k''}^{(1)} \hat{E}_{k'''}^{(1)} \rangle &= \mathcal{E}_k \mathcal{E}_{k''} \delta(k+k') \delta(k''+k''') \\ &+ \mathcal{E}_k \mathcal{E}_{k''} \delta(k+k''') \delta(k''+k') + \mathcal{E}_k \mathcal{E}_{k'} \delta(k+k'') \delta(k'+k''') . \end{aligned} \tag{c}$$ 1

In the Vlasov problem $E_k^{(1)}$ represents the k'th Fourier component of the first-order self-consistent electric field. In problems associated with the equation

$$\frac{\partial}{\partial t} \hat{E}_k = \int d\underline{k}' \, K(\underline{k}', \underline{k}-\underline{k}') \hat{E}_{k'} \hat{E}_{k-k'} e^{i(\omega_k - \omega_{k'} - \omega_{k-k'})t} , \tag{2}$$

the meaning of $\hat{E}_k$ depends on the physical problem being investigated (see for example reference 16).

In both cases the philosophy has been to solve E_k to a given order from the original dynamical equation, i.e.

$$\hat{E}_k \simeq \lambda \hat{E}_k^{(1)} + \lambda^2 \hat{E}_k^{(2)} + \lambda^3 \hat{E}_k^{(3)} + \cdots$$

Apart from initial conditions of $\hat{E}_k^{(2)}$ and $\hat{E}_k^{(3)}$ this gives $\hat{E}_k^{(2)}$ as an integral functional of the product of two $\hat{E}_k^{(1)}$'s and $\hat{E}_k^{(3)}$ as an integral functional of the product of $3\hat{E}_k^{(1)}$'s. Then a kinetic equation for the waves is obtained by performing averages of appropriate quantities over a statistical ensemble with the recipe given in Eq. (1).

We first comment that Eq. (1) is a reflection of the spatial homogeneity of the ensemble, that is to say the problems investigated have been ones of "homogeneous turbulence" 1(a) ensures that the autocorrelation function $\langle E(x)\,E(x+x_1)\rangle$ depends only on the relative co-ordinate x_1. 1(c) is analogous to V-17 and Eq. 12, Appendix H_3. In considering $\langle E_k^{(1)} E_{k'}^{(1)} E_{k''}^{(1)} E_{k'''}^{(1)}\rangle$, the taking of averages in all possible pairs corresponds to a cluster expansion (in this case for a spatially uniform ensemble) where the correlation of four waves is written in terms of products of correlations of 2 waves and the irreducible 4- correlations neglected.

Assumption 1(b) may or may not be true. In a more rigorous multiple time analysis of Eq. (2) as given in Appendix H_3, we see that the final kinetic equation for the waves describing the evolution of $\mathcal{E}_k$ is true <u>whether or not</u> 1(b) is satisfied.

APPENDIX I

Alternate Derivation of Hierarchy for Spatially Homogeneous Ensemble of Vlasov Fluids

Consider the Vlasov Equation

$$\frac{\partial f_{a_1}}{\partial t} + \underline{v}_1 \cdot \frac{\partial f_{a_1}}{\partial \underline{x}_1} = \frac{1}{m_{a_1}} \frac{\partial}{\partial \underline{v}_1} \cdot \sum_{a_2=1}^{\alpha} n_{a_2} \int \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} f_{a_1} f_{a_2} \, d(2) \,. \qquad 1$$

We write

$$f_{a_1} = \langle f_{a_1} \rangle + \Delta f_{a_1} \,,$$

where $\langle f_{a_1} \rangle$ is the ensemble average of f_{a_1} and is independent of $\underline{x}_1$.

Under the charge neutrality assumption

$$\sum_{a_2=1}^{\alpha} n_{a_2} e_{a_2} = 0 \,, \qquad 2$$

(1) becomes

$$\frac{\partial}{\partial t} \langle f_{a_1} \rangle + \left(\frac{\partial}{\partial t} \Delta f_{a_1} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} \Delta f_{a_1} - \frac{1}{m_{a_1}} \frac{\partial}{\partial \underline{v}_1} \langle f_{a_1} \rangle \cdot \sum_{a_2} n_{a_2} \int \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} \Delta f_{a_2} \, d(2) \right)$$

$$= \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \frac{\partial}{\partial \underline{v}_1} \cdot \int \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} \left(\Delta f_{a_1} \Delta f_{a_2} \right) d(2) \,, \qquad 3$$

With $\langle \Delta f_{a_1} \rangle = 0$, the ensemble average of (3) gives

$$\frac{\partial}{\partial t} \langle f_{a_1} \rangle = \frac{1}{m_{a_1}} \sum_{a_2} n_{a_2} \frac{\partial}{\partial \underline{v}_1} \cdot \int \frac{\partial \phi_{a_1 a_2}}{\partial \underline{x}_1} \langle \Delta f_{a_1} \Delta f_{a_2} \rangle \, d(2) \,. \qquad 4$$

The Fourier transform of (3) readily gives

$$\delta(\underline{k}_1)\frac{\partial}{\partial t}\langle f_{a_1}\rangle$$
$$+\left(\frac{\partial}{\partial t}\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)+i\underline{k}_1\cdot\underline{v}_1\,\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)\right.$$
$$\left.-\frac{4\pi e_{a_1}}{m_{a_1}}\frac{i\underline{k}}{k^2}\cdot\frac{\partial}{\partial\underline{v}_1}\langle f_{a_1}\rangle\sum_{a_2}n_{a_2}e_{a_2}\int d\underline{v}_2\,\Delta f_{a_2}(\underline{k}_2,\underline{v}_2)\right)$$

$$=\frac{i e_{a_1}}{m_{a_1}}\frac{\partial}{\partial\underline{v}_1}\cdot\sum_{a_2}4\pi n_{a_2}e_{a_2}\int d\underline{k}'\,\frac{\underline{k}-\underline{k}'}{|\underline{k}-\underline{k}'|^2}\int d\underline{v}_2\,\Delta f_{a_1}(\underline{k}',\underline{v}_1)\,\Delta f_{a_2}(\underline{k}-\underline{k}',\underline{v}_2). \qquad 5$$

We write as in Chapter V for a spatially homogeneous ensemble,

$$\langle\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)\Delta f_{a_2}(\underline{k}_2,\underline{v}_2,t)\rangle=\mathcal{G}_{a_1a_2}(\underline{k}_1,\underline{v}_1,\underline{v}_2,t)\,\delta(\underline{k}_1+\underline{k}_2), \qquad 6$$

$$\langle\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)\Delta f_{a_2}(\underline{k}_2,\underline{v}_2,t)\,\Delta f_{a_3}(\underline{k}_3,\underline{v}_3,t)\rangle \qquad 7$$
$$=\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3)\,\mathcal{H}_{a_1a_2a_3}(\underline{k}_1,\underline{k}_2,\underline{v}_1,\underline{v}_2,\underline{v}_3,t),$$

$$\langle\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)\Delta f_{a_2}(\underline{k}_2,\underline{v}_2,t)\,\Delta f_{a_3}(\underline{k}_3,\underline{v}_3,t)\,\Delta f_{a_4}(\underline{k}_4,\underline{v}_4,t)\rangle$$
$$=\sum_{\{1,2,3,4\}}\mathcal{G}_{a_1a_2}(\underline{k}_1,\underline{v}_1,\underline{v}_2,t)\,\mathcal{G}_{a_3a_4}(\underline{k}_3,\underline{v}_3,\underline{v}_4,t)\,\delta(\underline{k}_1+\underline{k}_2)\,\delta(\underline{k}_3+\underline{k}_4)$$
$$+K_{a_1a_2a_3a_4}(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{v}_1,\underline{v}_2,\underline{v}_3,\underline{v}_4,t)\,\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4). \qquad 8$$

Also $\qquad \langle\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)\rangle=0, \qquad 9$

which is consistent with $\langle\Delta f_{a_1}(\underline{x}_1,\underline{v}_1,t)\rangle=0$.

Equation 4 or the ensemble average of Equation 5 readily gives

$$\frac{\partial}{\partial t}\langle f_{a_1}(\underline{v}_1,t)\rangle=-i\frac{e_{a_1}}{m_{a_1}}\sum_{a_2}4\pi n_{a_2}e_{a_2}\int d\underline{k}\,d\underline{v}_2\,\frac{\underline{k}}{k^2}\cdot\frac{\partial}{\partial\underline{v}_1}\mathcal{G}_{a_1a_2}. \qquad 10$$

Introducing the operator

$$L_{a_1}(\underline{k}_1,\underline{v}_1,t) = i\underline{k}_1\cdot\underline{v}_1 - \frac{4\pi e_{a_1}}{m_{a_1}}\frac{i\underline{k}}{k^2}\cdot\frac{\partial}{\partial\underline{v}_1}\langle f_{a_1}\rangle\sum_{a_1} n_{a_1}e_{a_1}\int d\underline{v}_1\,, \tag{11}$$

we construct from Eq. (5) the following equation for $\left(\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)\,\Delta f_{a_2}(\underline{k}_2,\underline{v}_2,t)\right)$,

$$\left(\frac{\partial}{\partial t} + L_{a_1}(\underline{k}_1,\underline{v}_1,t) + L_{a_2}(\underline{k}_2,\underline{v}_2,t)\right)\left(\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)\,\Delta f_{a_2}(\underline{k}_2,\underline{v}_2,t)\right)$$
$$+\,\delta(\underline{k}_1)\,\Delta f_{a_2}(\underline{k}_2,\underline{v}_2,t)\,\frac{\partial}{\partial t}\langle f_{a_1}\rangle + \delta(\underline{k}_2)\,\Delta f_{a_1}(\underline{k}_1,\underline{v}_1,t)\,\frac{\partial}{\partial t}\langle f_{a_2}\rangle$$
$$+\left\{\frac{ie_{a_1}}{m_{a_1}}\sum_{a_3}4\pi n_{a_3}e_{a_3}\frac{\partial}{\partial\underline{v}_1}\cdot\int d\underline{k}'d\underline{v}_3\frac{\underline{k}_1-\underline{k}'}{|\underline{k}_1-\underline{k}'|^2}\,\Delta f_{a_1}(\underline{k}',\underline{v}_1,t)\,\Delta f_{a_3}(\underline{k}_1-\underline{k}',\underline{v}_3,t)\,\Delta f_{a_2}(\underline{k}_2,\underline{v}_2,t)\right.$$
$$\left. + (1\leftrightarrow 2)\right\}. \tag{12}$$

Averaging (12) according to (6), (7) and (9) yields

$$\left(\frac{\partial}{\partial t} + L_{a_1}(\underline{k}_1,\underline{v}_1,t) + L_{a_2}(\underline{k}_2,\underline{v}_2,t)\right)\mathcal{H}_{a_1a_2}(\underline{k}_1,\underline{v}_1,\underline{v}_2,t)\,\delta(\underline{k}_1+\underline{k}_2)$$
$$= \delta(\underline{k}_1+\underline{k}_2)\left\{\frac{4\pi ie_{a_1}}{m_{a_1}}\sum_{a_3}n_{a_3}e_{a_3}\frac{\partial}{\partial\underline{v}_1}\cdot\int d\underline{k}'d\underline{v}_3\,\frac{\underline{k}_1-\underline{k}'}{|\underline{k}_1-\underline{k}'|^2}\mathcal{H}_{a_1a_2a_3}(\underline{k}_1'-\underline{k}_1,\underline{v}_1,\underline{v}_2,\underline{v}_3,t)\right.$$
$$\left. + (1\leftrightarrow 2)\right\}. \tag{13}$$

Similarly for the product of 3 Δf_k's, we have

$$\left(\frac{\partial}{\partial t} + L_{a_1}(\underline{k}_1,\underline{v}_1,t) + L_{a_2}(\underline{k}_2,\underline{v}_2,t) + L_{a_3}(\underline{k}_3,\underline{v}_3,t)\right)\left(\Delta f_{a_1}\Delta f_{a_2}\Delta f_{a_3}\right)$$
$$+\sum_{\{1,2,3\}}\delta(\underline{k}_1)\,\Delta f_{a_2}(\underline{k}_2,\underline{v}_2,t)\,\Delta f_{a_3}(\underline{k}_3,\underline{v}_3,t)\,\frac{\partial}{\partial t}\langle f_{a_1}(v_1,t)\rangle$$
$$= \sum_{a_4}(4\pi n_{a_4}e_{a_4})\sum_{\{1,2,3\}}\frac{ie_{a_1}}{m_{a_1}}\int d\underline{k}'d\underline{v}_4\,\frac{\underline{k}_1-\underline{k}'}{|\underline{k}_1-\underline{k}'|^2}\,\Delta f_{a_1}(k_1',v_1)\,\Delta f_{a_2}(k_2,v_2)$$
$$\times\,\Delta f_{a_3}(k_3,v_3)\,\Delta f_{a_4}(\underline{k}_1-\underline{k}',\underline{v}_4)\;. \tag{14}$$

Averaging (14), utilizing (6), (7), (8) and Eq. (10) to eliminate for example

$$\langle \Delta f_{a_2}(\underline{k}_2, \underline{v}_2, t)\, \Delta f_{a_3}(\underline{k}_3, \underline{v}_3, t) \rangle \frac{\partial}{\partial t} \langle f_{a_1} \rangle$$

from the left side of the average of Eq. (14), gives

$$\left(\frac{\partial}{\partial t} + L_{a_1}(\underline{k}_1, \underline{v}_1, t) + L_{a_2}(\underline{k}_2, \underline{v}_2, t) + L_{a_3}(\underline{k}_3, \underline{v}_3, t) \right)$$

$$\times\, \mathcal{H}_{a_1 a_2 a_3}(\underline{k}_1, \underline{k}_2, \underline{v}_1, \underline{v}_2, \underline{v}_3, t)\, \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \qquad 15$$

$$= -\delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3) \sum_{a_4 = 1}^{\alpha} n_{a_4} e_{a_4} \sum_{\{1,2,3\}} \frac{4\pi i e_{a_1}}{m_{a_1}} \left\{ \int d\underline{v}_4 \frac{\underline{k}_3}{|\underline{k}_3|^2} \right.$$

$$\left. \cdot \frac{\partial}{\partial \underline{v}_1} G_{a_1 a_2}(-\underline{k}_2, \underline{v}_1, \underline{v}_2, t)\, G_{a_3 a_4}(\underline{k}_3, \underline{v}_3, \underline{v}_4, t) + (2 \leftrightarrow 3) \right\},$$

where we have truncated the system of equations by neglecting the K-correlations. Equations (10), (13) and (15) are identical to the system of equations II-25, II-26 and II-28, obtained from the B-B-G-K-Y hierarchy provided the source term S_k is neglected.

ACKNOWLEDGMENTS

It gives me great pleasure to thank Professor Edward Frieman for his constant advice and support during the course of this work. I am also indebted to Dr. Paul Rutherford for many helpful discussions. Thanks also must go to Dr. John Greene for reading the manuscript, Imperial Oil Company of Canada and the United States Atomic Energy Commission for financial aid during this research, and Mrs. Nona Cloutier for typing the manuscript.

Finally, I would like to thank my wife Jean, whose encouragement made possible the completion of this thesis.

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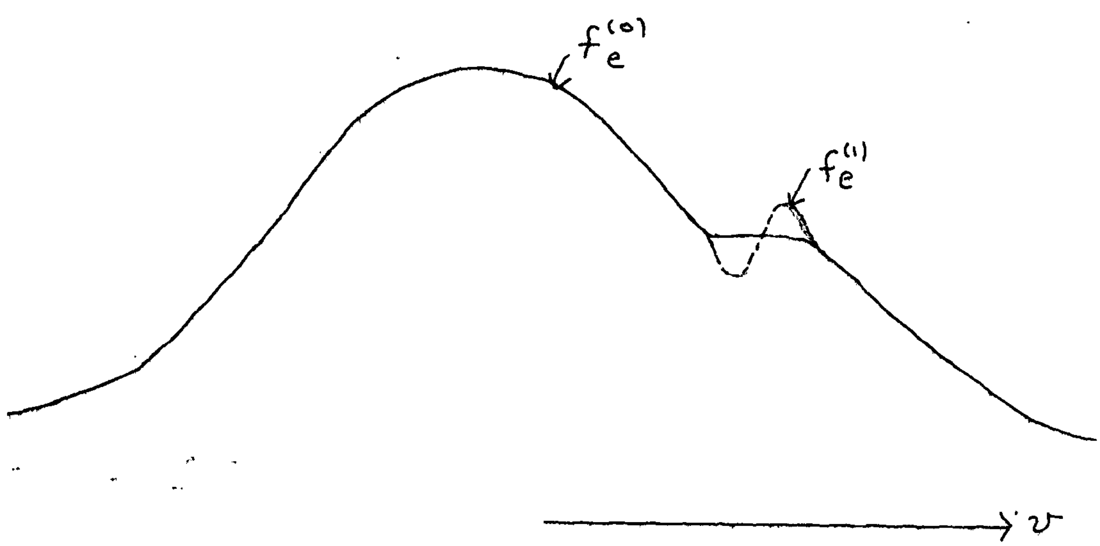

DIAGRAM 1: Electron velocity distribution function for slightly unstable one-dimensional plasma

WEAK TURBULENCE IN A HOMOGENEOUS PLASMA

ABSTRACT

The problem of weak plasma turbulence is considered within the framework of the B-B-G-K-Y hierarchy of equations for a fully ionized, magnetic field-free, spatially homogeneous, ensemble of slightly unstable (in velocity space), multispecies, plasmas. The hierarchy equations, utilizing the Bogoliubov-Krylov multiple time-scale technique, are expanded in the small parameter ϵ_q where

$$1 \gg \epsilon_q \sim \frac{\psi_{\underline{k}}}{n m v_{av}^2} \sim \frac{\gamma_{\underline{k}}}{\omega_{\underline{k}}} \gg \epsilon_p \quad .$$

The quantity ψ_k is the wave energy density associated with the unstable modes; $n m v_{av}^2$ represents the particle kinetic energy density; $\gamma_{\underline{k}}$ is a linear growth rate typical of the unstable modes and, ω_k the associated oscillation frequency. The quantity ϵ_p is the usual plasma parameter $(1/\epsilon_p \sim$ the number of particles in a Debye sphere). In the analysis, which is carried out to order ϵ_q^2, closure of the hierarchy equations is obtained by ordering out the irreducible four-particle correlations.

The final equations advance in time, the one-particle distribution function f(1), and the wave energy density $\psi_{\underline{k}}$. Included are the nonlinear effects of wave-wave interactions as well as the linear and nonlinear effects of wave-particle scattering. The effect of particle-particle encounters is considered in reference to the kinetic equations for $\psi_{\underline{k}}$.

Various properties of the final kinetic equations for f(1) and $\psi_{\underline{k}}$ are considered. These include a demonstration of particle momentum conservation, wave energy plus particle kinetic energy conservation, and also that if the spectrum $\psi_{\underline{k}}$ is initially positive, it never turns negative (a result necessary

for any stabilization arguments). The properties of the wave-wave terms and conditions for their validity are discussed in some detail. In the special case of a one-dimensional electron plasma the concept of quasi-particles is introduced and quasi-particle conservation laws associated with the wave-wave terms are demonstrated; in addition, stabilization ($\gamma_k \rightarrow 0$) of the distribution function is shown. The rapidity of decay of velocity moments of functions over typical free-streaming, $e^{-i\underline{k}\cdot\underline{v}t}$, terms is also investigated. A lower bound on the k-values allowed in the theory and a condition for neglect of particle trapping are obtained.

In a separate example a simple model equation exhibiting solely wave-wave interactions, is considered. The determining equation is of such a form that in the lowest-order, linearized version, waves of different wave numbers propagate independently. To next order, the nonlinear wave interactions act as perturbations which slowly transfer energy between modes. Two rigorous techniques of solution are developed. In one approach the coherent problem of solving for the wave amplitude order by order is treated; then, by performing a suitable statistical average over a spatially homogeneous ensemble, a kinetic equation for the wave energy density is obtained. In the other approach, equations for wave correlations characterizing the ensemble are constructed at the outset from the original dynamical equation. From these, a kinetic equation describing the evolution of the wave energy density is obtained. The latter approach is a very direct way to do the problem and results in major algebraic and conceptional simplifications compared to the former approach. This is a manifestation of the fact that the coherent problem entails much more information (and hence algebra) than is necessary for describing the ensemble.

The principle of ensemble averaging at an early stage (in lieu of solving the coherent problem order by order and then averaging) is considered in reference to the Vlasov-Poisson equations in the problem of weak turbulence in a plasma. It is shown that this approach leads to equations for correlations, identical to the B-B-G-K-Y hierarchy if the particle-particle terms associated with the discreteness of matter are omitted from the latter formalism. This result gives strong motivation for using the hierarchy equations in the problem of plasma turbulence instead of using the technique prevalent in the literature of solving the coherent Vlasov problem order by order and then averaging.

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