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An energy principle for ideal MHD equilibria with flows

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In the standard ideal MHD energy principle for equilibria with no flows, the stability criterion, which is the definiteness of the perturbed potential energy, is usually constructed from the linearized equation of motion. Equivalently while more straightforwardly, it can also be obtained from the second variation of the Hamiltonian calculated with proper constraints. For equilibria with flows, a stability criterion was proposed from the linearized equation of motion, but not explained as an energy principle[1]. In this paper, the second variation of the Hamiltonian is found to provide a stability criterion equivalent to, while more straightforward than, what was constructed from the linearized equation of motion. To calculate the variations of the Hamiltonian, a complete set of constraints on the dynamics of the perturbations is derived from the Euler-Poincare structure of the ideal MHD. In addition, a previous calculation of the second variation of the Hamiltonian was claimed to give a different stability criterion[2], and in this paper we argue such a claim is incorrect.

I. INTRODUCTION

The stability of ideal MHD equilibria is of great concern in fusion, astro and space plasma physics. For equilibria with no flows, a well developed tool to study the stability is the energy principle[3]. Traditionally, the stability criterion is constructed from the linearized equation of motion $\rho\ddot{\boldsymbol{\xi}} = \mathbf{F}_0(\boldsymbol{\xi})$, where ρ is the mass density, $\boldsymbol{\xi}$ is the displacement and $\mathbf{F}_0(\boldsymbol{\xi})$ is the force operator for ideal MHD equilibria with no flows[3]. It is found that $\boldsymbol{\xi}$ has no positive growth rate when $-\int \boldsymbol{\xi}\cdot\mathbf{F}_0(\boldsymbol{\xi})\,\mathrm{d}^3x$, which turns out to be the perturbed potential energy $\delta^2 W_0$, is positive definite. With such an approach one needs to prove the self-adjointness of the force operator $\mathbf{F}_0(\boldsymbol{\xi})$, that $\int \boldsymbol{\eta}\cdot\mathbf{F}_0(\boldsymbol{\xi})\,\mathrm{d}^3x = \int \boldsymbol{\xi}\cdot\mathbf{F}_0(\boldsymbol{\eta})\,\mathrm{d}^3x$, which requires tedious algebra. From an equivalent but more straightforward perspective, the perturbed energy is in fact the second variation of the total Hamiltonian calculated with proper variational constraints, $\delta^2 H = \int \rho\dot{\boldsymbol{\xi}}^2\,\mathrm{d}^3x + \delta^2 W_0$. As the total energy is conserved in ideal MHD, the perturbed energy is intrinsically conserved. As a result, the positive definiteness of $\delta^2 W_0$ can be proven to be a necessary and sufficient condition for the equilibrium to be stable[4]. Moreover, the self-adjointness of the force operator can be proven as a result of the conservation of the perturbed energy[5].

The stability of ideal MHD equilibria with non-zero flows is also important, especially for fusion plasmas where macroscopic flows are ubiquitous[6]. A stability criterion was previously constructed by Frieman and Rotenberg[1] from the linearized equation of motion with flows

$$\rho\ddot{\boldsymbol{\xi}} + 2\rho\mathbf{v}\cdot\nabla\dot{\boldsymbol{\xi}} = \mathbf{F}(\boldsymbol{\xi}), \tag{1}$$

where $\mathbf{v}$ is the non-zero equilibrium flow velocity, and here the force operator

$$\mathbf{F}(\boldsymbol{\xi}) = \nabla\cdot(\rho\boldsymbol{\xi}\mathbf{v}\cdot\nabla\mathbf{v} - \rho\mathbf{v}\mathbf{v}\cdot\nabla\boldsymbol{\xi}) + \mathbf{F}_0(\boldsymbol{\xi}) \tag{2}$$

has two more flow-dependent terms than $\mathbf{F}_0(\boldsymbol{\xi})$. They found that when $-\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\,\mathrm{d}^3x$ is positive definite, $\boldsymbol{\xi}$ has no positive growth rate, and therefore the system is stable. Although this criterion turns out valid, as will be rederived from the Hamiltonian perspective in this paper, it has several defects. First, the criterion was not explained in an energy perspective, and hence was not called as an energy principle. Secondly, one needs to prove the self-adjointness of the flow-dependent force operator $\mathbf{F}(\boldsymbol{\xi})$, which now requires even more algebra because of the non-zero flows, while in principle it should be a natural outcome of the conservation of the perturbed energy.

In this paper, from the Hamiltonian perspective, we present an energy principle that is equivalent to Frieman and Rotenberg's stability criterion. We derive a complete set of variational constraints on the perturbations that are allowed by their dynamics, which are derived from the Euler-Poincare structure[7,8] of the ideal MHD. With these variational constraints, the intrinsically conserved second variation of the Hamiltonian $\delta^2 H = \int \rho \dot{\boldsymbol{\xi}}^2 \,\mathrm{d}^3x + \delta^2 W$ leads to an energy principle that the system is stable when the effective potential energy $\delta^2 W = -\int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}) \,\mathrm{d}^3x$ is positive definite. Such a stability criterion is found equivalent to the one constructed from the linearized equation of motion, but derived more straightforwardly.

It has come to our attention that a previous calculation of the second variation of the Hamiltonian was claimed to give a different stability criterion[2]. To summarize the result, the so-called "dynamically accessible" variations, as introduced by Isichenko[9] and Morrison[10], were used as constraints by Hameiri[2] to obtain the second variation of the Hamiltonian, $\delta^2 H = \int [\rho(\delta\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla\mathbf{v})^2 - \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})] \,\mathrm{d}^3x$, the definiteness of which was claimed to determine the stability of the equilibrium. Note that here $\delta\mathbf{v}$ is of the dynamically accessible form expressed with variables $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$,

$$\delta\mathbf{v} = \boldsymbol{\xi} \times (\nabla \times \mathbf{v}) + (\nabla \times \boldsymbol{\zeta}) \times \mathbf{B}/\rho + \alpha\nabla s - \nabla\beta, \tag{3}$$

where s and $\mathbf{B}$ are the specific entropy and the magnetic flux density respectively. The arbitrariness of the variables $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$ allows for manipulations for stability analysis, therefore the resulted stability criterion was claimed to be different from, and better than what Frieman and Rotenberg proposed, which is surprising since intuitively the stability criterion derived from the two perspectives are expected to be equivalent. We believe such a criterion is incorrect for the following reasons. First, the variational constraints they used were not rigorously derived. Therefore, their interpretation that the variational constraints are equivalent to the Casimirs[2,10] may not stand. In this paper, the "dynamically accessible" variations are recovered in our constraints, and for the first time, rigorously derived. In the derivation the dynamically accessible form of $\delta\mathbf{v}$ is found to be a result of the Euler-Poincare structure of ideal MHD, rather than a conservation law. Secondly, the dynamics of the perturbations were not considered, while in this paper it is shown that the variables $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$ in the dynamically accessible form of $\delta\mathbf{v}$ are in fact constrained by some dynamical equations, and hence not arbitrary variables as claimed in Ref. 2. Thirdly, the reason why the definiteness of $\delta^2 H$ determines the stability of the equilibrium was argued to be a mixture[10] of the energy-Casimir method[11,12] and the Dirichlet's theorem[12], which has not really been proven valid for noncanonical infinite-dimensional systems like ideal MHD[10]. We believe it should be the conservation of $\delta^2 H$, which is a dynamical effect, that leads to the correct stability criterion.

The rest of the paper is arranged as follows. By studying the Lagrangian formulation of the ideal MHD in Sec. II, and its Euler-Poincare structure in Sec. III, we obtain a complete set of variational constraints that describes the dynamics of the ideal MHD perturbations in Sec. IV. In Sec. V, the constraints are then used to calculate the second variation of the Hamiltonian and develop a stability criterion, which is further shown to be equivalent to the one obtained by Frieman and Rotenberg. The arguments against Hameiri's criterion and a brief conclusion are presented in Sec. VI.

II. THE LAGRANGIAN FORMULATION OF THE IDEAL MHD

To establish the Lagrangian formulation of the ideal MHD, we start with the action principle $\delta \int L[\mathbf{v}, \rho, s, \mathbf{B}]\,\mathrm{d}t = 0$, where L is the Lagrangian

$$L = \int \left[\frac{1}{2}\rho\mathbf{v}^2 - \rho\epsilon(s,\rho) - \frac{1}{2}\mathbf{B}^2\right]\mathrm{d}^3x, \tag{4}$$

and the specific internal energy ϵ is defined by the first law of thermodynamics $\mathrm{d}\epsilon = T\mathrm{d}s - p\mathrm{d}(1/\rho)$, where the pressure p and the temperature T are also functions of ρ and s. After directly trying with $\{\delta\mathbf{v}, \delta\rho, \delta s, \delta\mathbf{B}\}$ assumed to be arbitrary perturbations, one finds that calculating $\delta \int L[\mathbf{v}, \rho, s, \mathbf{B}]\,\mathrm{d}t = 0$ fails to give the expected equation of motion. This suggests that appropriate variational constraints on $\{\delta\mathbf{v}, \delta\rho, \delta s, \delta\mathbf{B}\}$ must be used.

Variational constraints are often connected with conservation laws, and in ideal MHD we have the continuity law, the adiabatic law, and the frozen-in law, which correspond to the local conservation of mass, entropy, and magnetic flux respectively. The idea of local conservation is that certain quantities associated with an arbitrary fluid element are carried along by the flow. Take the conservation of mass $M = \int_V \rho\mathrm{d}^3x$ as an example, where $V(t)$ is an arbitrary volume that flows with the fluid. Perturbed by a small displacement $\boldsymbol{\xi}$, the mass within the volume remains constant,

$$\delta M = \int_V \delta\rho\mathrm{d}^3x + \oint_{\partial V} \rho\boldsymbol{\xi}\cdot\mathrm{d}\mathbf{S} = \int_V [\delta\rho + \nabla\cdot(\rho\boldsymbol{\xi})]\mathrm{d}^3x = 0. \tag{5}$$

Now that V is arbitrary, one has

$$\delta\rho = -\nabla \cdot (\rho\boldsymbol{\xi}). \tag{6}$$

Such a proof is inspired by Arnold[13]. Similarly, from the conservation of entropy and magnetic flux, we have

$$\delta s = -\boldsymbol{\xi} \cdot \nabla s, \tag{7}$$

$$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}). \tag{8}$$

One would recognize that Eqs. (6) - (8) look exactly the same as the constraints used in the standard energy principle with no flows[3], which were obtained by linearizing and integrating the ideal MHD equations when $\mathbf{v} = 0$. However, they are now proven valid even when there is non-zero flow. These variational constraints can make sure that the local conservation of mass, entropy, and magnetic flux is not violated.

As for the velocity $\mathbf{v}$, there is not a similar local conservation law associated, but $\delta\mathbf{v}$ is related to the displacement $\boldsymbol{\xi}$ in a different way. Express $\boldsymbol{\xi}$ with the Lagrangian labeling X, $\boldsymbol{\xi}(\mathbf{x}, t) = \delta\mathbf{x}(X, t)$, and then take the time derivative, we have $\dot{\boldsymbol{\xi}} + \mathbf{v} \cdot \nabla\boldsymbol{\xi} = \delta\dot{\mathbf{x}}$. Similarly, $\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}}(X, t)$, and by taking the variation we get $\delta\mathbf{v} + \boldsymbol{\xi} \cdot \nabla\mathbf{v} = \delta\dot{\mathbf{x}}$. Combining the two equations leads to a constraint on $\delta\mathbf{v}$,

$$\delta\mathbf{v} = \dot{\boldsymbol{\xi}} + \mathbf{v} \cdot \nabla\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla\mathbf{v}. \tag{9}$$

With these variational constraints, the action principle $\delta \int L \, \mathrm{d}t = 0$ now gives the momentum equation,

$$\partial_t(\rho\mathbf{v}) = -\nabla \cdot (\rho\mathbf{v}\mathbf{v}) + (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p. \tag{10}$$

Meanwhile, the other ideal MHD equations directly come from the local conservation of mass, entropy, and magnetic flux respectively,

$$\partial_t\rho = -\nabla \cdot (\rho\mathbf{v}), \tag{11}$$

$$\partial_t s = -\mathbf{v} \cdot \nabla s, \tag{12}$$

$$\partial_t\mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{13}$$

Such a formulation was first presented by Newcomb[14]. The Hamiltonian is defined by

$$H[\mathbf{v}, \rho, s, \mathbf{B}] = \int \frac{\delta L}{\delta\mathbf{v}} \cdot \mathbf{v} \, \mathrm{d}^3x - L = \int \left[\frac{1}{2}\rho\mathbf{v}^2 + \rho\epsilon(s, \rho) + \frac{1}{2}\mathbf{B}^2\right] \mathrm{d}^3x. \tag{14}$$

Using the ideal MHD equations, one can easily prove that H is conserved. When the system is at equilibrium, one would expect the first variation of the Hamiltonian δH to vanish, just like what happens with equilibrium with no flows[5]. However, with the variational constraints (6) - (9) and the equilibrium conditions, δH does not vanish. One also notices that the constraints (6) - (9) does not form a closed dynamical system. That is, $\boldsymbol{\xi}$ cannot be solved self-consistently from the four equations. This suggests that there should be more constraints on the dynamics of the perturbations.

III. THE EULER-POINCARE REDUCTION

The formulation in the last section is in fact the ideal MHD case of the Euler-Poincare reduction[7]. In this section, the language of the Euler-Poincare reduction for general continua is introduced to uncover the geometric structure of the ideal MHD, which is buried under the massive vector algebra. More detailed discussion on the Euler-Poincare reduction can be found in Refs. 7 and 8.

Describe a continuum with the Lagrangian labeling X, and let the Lagrangian be $L[\mathbf{x}(X,t), \dot{\mathbf{x}}(X,t), a_0(X)]$, where $a_0(X)$ is the advected quantity that is frozen into the fluid element when the continuum flows. The fluid motion is captured by a path η_t in the configuration space Diff($\mathcal{D}$), which maps the Lagrangian labeling into the Eulerian labeling: $\eta_t(X) = \mathbf{x}(X,t)$. The action principle $\delta \int L \,\mathrm{d}t = 0$ then gives the Euler-Lagrange equation for η_t on Diff($\mathcal{D}$) in the Lagrangian labeling.

However, one would prefer to use the more familiar Eulerian labeling: the Lagrangian $l[\mathbf{v}(\mathbf{x},t), a(\mathbf{x},t)] = L[\mathbf{x}, \dot{\mathbf{x}}, a_0] \circ \eta_t^{-1}$ lives on the semidirect product space $\mathfrak{X}(\mathcal{D}) \times V^*$, where $\mathbf{v}(\mathbf{x},t) = \dot{\mathbf{x}} \circ \eta_t^{-1} \in \mathfrak{X}(\mathcal{D})$, the Lie algebra, and $a(\mathbf{x},t) = a_0 \circ \eta_t^{-1} \in V^*$, the representation space. To deal with the action principle in Eulerian labeling $\delta \int l[\mathbf{v}, a] \,\mathrm{d}t = 0$, one needs the constraints on $\delta\mathbf{v}$ and δa,

$$\delta \mathbf{v} = \dot{\boldsymbol{\xi}} + \mathbf{v} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \mathbf{v}, \tag{15}$$

$$\delta a = -\mathfrak{L}_{\boldsymbol{\xi}} a, \tag{16}$$

where $\boldsymbol{\xi}(\mathbf{x},t) = \delta\mathbf{x} \circ \eta_t^{-1} \in \mathfrak{X}(\mathcal{D})$, and the Lie derivative $\mathfrak{L}_{\boldsymbol{\xi}} a = a \circ \delta\eta_t \circ \eta_t^{-1}$. The exact form of $\mathfrak{L}_{\boldsymbol{\xi}} a$ is determined case by case depending on the nature of the tensor a. Eq. (15) is Eq. (9) and has been derived in Sec. II, while Eq. (23) is obtained by taking the variation of

$a \circ \eta_t = a_0$. With these constraints the action principle leads to the Euler-Poincare equation for continua

$$\partial_t \frac{\delta l}{\delta \mathbf{v}} = -\mathfrak{L}_{\mathbf{v}} \frac{\delta l}{\delta \mathbf{v}} + \frac{\delta l}{\delta a} \diamond a, \tag{17}$$

where the $\diamond$ operation is defined by $\langle w \diamond a, \mathbf{u} \rangle = -\langle \mathfrak{L}_{\mathbf{u}} a, w \rangle$, for all $w \in V$, $a \in V^*$, $\mathbf{u} \in \mathfrak{X}(\mathcal{D})$, and $\langle \, , \, \rangle = \int \mathrm{d}^3 x$. The rest of the dynamical equations is simply the advection equation,

$$\partial_t a = -\mathfrak{L}_{\mathbf{v}} a. \tag{18}$$

In addition, the Hamiltonian is defined by $h[\mathbf{v}, a] = \langle \delta l / \delta \mathbf{v}, \mathbf{v} \rangle - l$, which can be proven to be conserved using Eqs. (17) and (18). It is worthwhile to clarify that here h is defined in $\mathfrak{X}(\mathcal{D}) \times V^*$ with variables $\{\mathbf{v}, a\}$, while usually the Hamiltonian formulation is developed in $\mathfrak{X}^*(\mathcal{D}) \times V^*$ with the Legendre transformation. We choose to stay in $\mathfrak{X}(\mathcal{D}) \times V^*$ because the role the Hamiltonian plays in our energy principle is no more than a conserved quantity, so there is no need to confuse the readers by defining more variables and switching spaces.

In the ideal MHD, $\delta l / \delta \mathbf{v} = \rho \mathbf{v}$, while $a \in \{\rho \, \mathrm{d}^3 x, s, \mathbf{B} \cdot \mathrm{d}\mathbf{S}\}$, and $\mathfrak{L}_{\mathbf{v}}(\rho \, \mathrm{d}^3 x) = \nabla \cdot (\rho \mathbf{v}) \, \mathrm{d}^3 x$, $\mathfrak{L}_{\mathbf{v}} s = \mathbf{v} \cdot \nabla s$, and $\mathfrak{L}_{\mathbf{v}}(\mathbf{B} \cdot \mathrm{d}\mathbf{S}) = -\nabla \times (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S}$. Then Eq. (16) and Eq. (18) become Eqs. (6) - (8) and Eqs. (11) - (13) respectively, while Eq. (15) is Eq. (9), and Eq. (17) becomes

$$\partial_t(\rho \mathbf{v}) = -\nabla \cdot (\mathbf{v} \rho \mathbf{v}) - \nabla \mathbf{v} \cdot \rho \mathbf{v} + \rho \nabla (\mathbf{v}^2/2 - \epsilon - p/\rho) + \rho T \nabla s + (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{19}$$

which can be simplified to Eq. (10) after some algebra.

In the next section, the language of the Euler-Poincare reduction will be further used to derive the complete set of constraints on the linear perturbations.

IV. THE VARIATIONAL CONSTRAINTS

It is noticed that the constraints (24) - (23) does not form a closed dynamical system from which the evolution of $\boldsymbol{\xi}$ can be solved self-consistently. This suggests there should be more constraints, which we will derive by studying the dynamics of the perturbations in this section. First consider the perturbations $\{\delta \mathbf{v}, \delta a\} \in \mathfrak{X}(\mathcal{D}) \times V^*$. In principle, the dynamics of any linear perturbation $\{\delta \mathbf{v}, \delta a\}$ must follow the linearized equations of motion for the continuum:

$$\partial_t \left(\delta \frac{\delta l}{\delta \mathbf{v}} \right) = -\mathfrak{L}_{\delta \mathbf{v}} \frac{\delta l}{\delta \mathbf{v}} - \mathfrak{L}_{\mathbf{v}} \left(\delta \frac{\delta l}{\delta \mathbf{v}} \right) + \left(\delta \frac{\delta l}{\delta a} \right) \diamond a + \frac{\delta l}{\delta a} \diamond \delta a, \tag{20}$$

$$\partial_t \delta a = -\mathfrak{L}_{\delta \mathbf{v}} a - \mathfrak{L}_{\mathbf{v}} \delta a. \tag{21}$$

Note that $\delta l/\delta \mathbf{v}$ and $\delta l/\delta a$ can be expressed with $\{\mathbf{v}, a\}$, therefore $\delta(\delta l/\delta \mathbf{v})$ and $\delta(\delta l/\delta a)$ can be expressed with $\{\delta\mathbf{v}, \delta a\}$. The equations above form a closed dynamical system with variables $\{\delta\mathbf{v}, \delta a\}$.

With the equations above constraining their dynamics, $\{\delta\mathbf{v}, \delta a\}$ must not be arbitrary, but in certain forms. We have found that $\{\delta\mathbf{v}, \delta a\}$ must be in the following algebraic form

$$\delta\frac{\delta l}{\delta \mathbf{v}} = -\mathfrak{L}_{\boldsymbol{\xi}}\frac{\delta l}{\delta \mathbf{v}} + a^* \diamond a, \tag{22}$$

$$\delta a = -\mathfrak{L}_{\boldsymbol{\xi}} a, \tag{23}$$

expressed with variables $\{\boldsymbol{\xi}, a^*\} \in \mathfrak{X}(\mathcal{D}) \times V$ (again, note that $\delta(\delta l/\delta \mathbf{v})$ and $\delta(\delta l/\delta a)$ can be expressed with $\{\delta\mathbf{v}, \delta a\}$). And in order for $\{\delta\mathbf{v}, \delta a\}$ to satisfy Eqs. (20) and (21), the evolution of $\{\boldsymbol{\xi}, a^*\}$ must follow the differential equations

$$\partial_t\boldsymbol{\xi} = \delta\mathbf{v} + \boldsymbol{\xi}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\xi}, \tag{24}$$

$$\partial_t a^* = \delta\frac{\delta l}{\delta a} + \mathfrak{L}_{\boldsymbol{\xi}}\frac{\delta l}{\delta a} - \mathfrak{L}_{\mathbf{v}} a^*, \tag{25}$$

where $\delta\mathbf{v}$ and $\delta(\delta l/\delta a)$ can be expressed with $\{\boldsymbol{\xi}, a^*\}$, using Eqs. (22) and (23). The four equations (22) - (25) form a closed dynamical system that describes the dynamics of the perturbations with the variables $\{\boldsymbol{\xi}, a^*\}$: Eqs. (22) and (23) change the variables from $\{\delta\mathbf{v}, \delta a\}$ to $\{\boldsymbol{\xi}, a^*\}$, while Eqs. (24) and (25) describe the dynamics of the new variables $\{\boldsymbol{\xi}, a^*\}$. By combining the four equations one gets Eqs. (20) and (21), which suggests that the two dynamical systems are indeed equivalent.

The four equations (22) - (25) are the complete set of constraints on the perturbations $\{\delta\mathbf{v}, \delta a\}$ that will be further used to calculate the variations of the Hamiltonian, but before doing that we must prove the validity of them. Eqs. (23) and (24) are just constraints (16) and (15), which have been proved in Sec. III. Next, we shall prove Eqs. (22) and (25).

First, define a vector field $\mathbf{u}_0(X) \in T_{\eta_t(X)}D$ that is carried by the continuum flow η_t, and then express it with the Eulerian labeling $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0 \circ \eta_t^{-1} \in \mathfrak{X}(\mathcal{D})$. Then, pair it with the Euler-Lagrange equation Eq. (17), we have

$$\left\langle (\partial_t + \mathfrak{L}_{\mathbf{v}})\frac{\delta l}{\delta \mathbf{v}}, \mathbf{u} \right\rangle = \left\langle \frac{\delta l}{\delta a} \diamond a, \mathbf{u} \right\rangle = -\left\langle \frac{\delta l}{\delta a}, \mathfrak{L}_{\mathbf{u}} a \right\rangle. \tag{26}$$

The last equal sign comes from the definition of the $\diamond$ operator. Then, map Eq. (26) to the Lagrangian labeling with η_t,

$$\left\langle \left[(\partial_t + \mathfrak{L}_{\mathbf{v}})\frac{\delta l}{\delta \mathbf{v}}\right] \circ \eta_t, \mathbf{u}_0 \right\rangle = -\left\langle \frac{\delta L}{\delta a_0}, (\mathfrak{L}_{\mathbf{u}} a) \circ \eta_t \right\rangle. \tag{27}$$

For the LHS, we use the property of the material derivative (derivation along a Co-Adjoint orbit)[8]

$$\left[(\partial_t + \mathfrak{L}_{\mathbf{v}})\frac{\delta l}{\delta \mathbf{v}}\right] \circ \eta_t = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\delta L}{\delta \dot{\mathbf{x}}}\right) \circ \eta_t\right] = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{\mathbf{x}}}. \tag{28}$$

Meanwhile, for the RHS, a property of the Lie derivative is[12]

$$(\mathfrak{L}_{\mathbf{u}} a) \circ \eta_t = \mathfrak{L}_{\mathbf{u}\circ\eta_t} a \circ \eta_t = \mathfrak{L}_{\mathbf{u}_0} a_0. \tag{29}$$

As a result, Eq. (27) becomes

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{\mathbf{x}}}, \mathbf{u}_0 \right\rangle = -\left\langle \frac{\delta L}{\delta a_0}, \mathfrak{L}_{\mathbf{u}_0} a_0 \right\rangle. \tag{30}$$

Then, define $a_0^*(X, t)$ with

$$\frac{\mathrm{d}a_0^*}{\mathrm{d}t} = \delta \frac{\delta L}{\delta a_0}. \tag{31}$$

After integrating and then varying Eq. (30) (note that a_0 and $\mathbf{u}_0$ are fixed), one gets

$$\left\langle \delta \frac{\delta L}{\delta \dot{\mathbf{x}}}, \mathbf{u}_0 \right\rangle = -\left\langle a_0^*, \mathfrak{L}_{\mathbf{u}_0} a_0 \right\rangle. \tag{32}$$

Mapping Eq. (32) back to the Eulerian labeling with η_t^{-1}, we have

$$\left\langle \left(\delta \frac{\delta L}{\delta \dot{\mathbf{x}}}\right) \circ \eta^{-1}, \mathbf{u} \right\rangle = -\left\langle a^*, (\mathfrak{L}_{\mathbf{u}_0} a_0) \circ \eta_t^{-1} \right\rangle = -\left\langle a^*, \mathfrak{L}_{\mathbf{u}} a \right\rangle, \tag{33}$$

where $a^*(\mathbf{x}, t) = a_0^* \circ \eta_t^{-1}$, and on the RHS the inverse of Eq. (29) is used. For the LHS, we use the property of the material variation, the proof of which is similar with Eq. (28),

$$\delta \frac{\delta L}{\delta \dot{\mathbf{x}}} = \delta \left[\left(\frac{\delta L}{\delta \dot{\mathbf{x}}}\right) \circ \eta_t\right] = \left[(\delta + \mathfrak{L}_{\boldsymbol{\xi}})\frac{\delta l}{\delta \mathbf{v}}\right] \circ \eta_t, \tag{34}$$

then Eq. (33) becomes

$$\left\langle (\delta + \mathfrak{L}_{\boldsymbol{\xi}})\frac{\delta l}{\delta \mathbf{v}}, \mathbf{u} \right\rangle = -\left\langle a^*, \mathfrak{L}_{\mathbf{u}} a \right\rangle = \left\langle a^* \diamond a, \mathbf{u} \right\rangle. \tag{35}$$

Since $\mathbf{u}$ can be arbitrary, we have one constraint, Eq. (22). Meanwhile, similar with Eqs. (28) and (34), we have

$$\frac{\mathrm{d}a_0^*}{\mathrm{d}t} = [(\partial_t + \mathfrak{L}_{\mathbf{v}})a^*] \circ \eta_t, \; \delta \frac{\delta L}{\delta a_0} = \left[(\delta + \mathfrak{L}_{\boldsymbol{\xi}})\frac{\delta l}{\delta a}\right] \circ \eta_t, \tag{36}$$

and therefore Eq. (31) turns into the other constraint, Eq. (25).

So now we have used the new variables $\{\boldsymbol{\xi}, a^*\}$ instead of $\{\delta \mathbf{v}, \delta a\}$ to describe the perturbations, but are the new variables any better than the old ones? As it turns out, they are more useful since the algebraic constraints (22) and (23) capture the Euler-Poincare structure of the system. One evidence is that, with Eqs. (22) and (23), the first variation of the Hamiltonian

$$\begin{aligned}\delta h &= \left\langle \delta\frac{\delta l}{\delta \mathbf{v}}, \mathbf{v}\right\rangle + \left\langle \frac{\delta l}{\delta \mathbf{v}}, \delta\mathbf{v}\right\rangle - \left\langle \frac{\delta l}{\delta a}, \delta a\right\rangle - \left\langle \frac{\delta l}{\delta \mathbf{v}}, \delta\mathbf{v}\right\rangle \\ &= \left\langle \left(\mathfrak{L}_{\mathbf{v}}\frac{\delta l}{\delta \mathbf{v}} - \frac{\delta l}{\delta a}\diamond a\right), \boldsymbol{\xi}\right\rangle - \langle \mathfrak{L}_{\mathbf{v}} a, a^*\rangle \end{aligned} \tag{37}$$

is shown to automatically vanish when the system is at equilibrium, where the LHS of Eqs. (17) and (18) both equal zero, just as expected.

To apply the above results to the ideal MHD, where $a \in \{s, \rho\, \mathrm{d}^3x, \mathbf{B}\cdot \mathrm{d}\mathbf{S}\}$, one defines

$$\rho^* = \mathbf{v}\cdot\boldsymbol{\xi} - \beta,\; s^* = -\rho\alpha,\; \mathbf{B}^* = -\boldsymbol{\zeta}, \tag{38}$$

then Eq. (22) becomes

$$\delta\mathbf{v} = \boldsymbol{\xi}\times(\nabla\times\mathbf{v}) + (\nabla\times\boldsymbol{\zeta})\times\mathbf{B}/\rho + \alpha\nabla s - \nabla\beta, \tag{39}$$

which is the same as Eq. (3), and Eq. (25) becomes

$$\dot\beta + \mathbf{v}\cdot\nabla\beta = \delta p/\rho + \boldsymbol{\xi}\cdot(\nabla\times\mathbf{B})\times\mathbf{B}/\rho, \tag{40}$$

$$\dot\alpha + \mathbf{v}\cdot\nabla\alpha = \delta T + \boldsymbol{\xi}\cdot\nabla T, \tag{41}$$

$$\dot{\boldsymbol{\zeta}} + \mathbf{v}\cdot\nabla\boldsymbol{\zeta} + \nabla\mathbf{v}\cdot\boldsymbol{\zeta} = \delta\mathbf{B} + \boldsymbol{\xi}\cdot\nabla\mathbf{B} + \nabla\boldsymbol{\xi}\cdot\mathbf{B}. \tag{42}$$

Eq. (39) was first introduced by Isichenko[9]. Together with Eqs. (6) - (8), it was also obtained by Morrison[10] from the non-canonical Poisson bracket and referred to as the "dynamically accessible" variations. However, neither of them provided detailed derivation. Here we have rigorously derived it from the Euler-Poincare reduction, and meanwhile presented Eqs. (40) - (42) which describe the evolution of $\{\beta, \alpha, \boldsymbol{\zeta}\}$, for the first time. From Eqs. (9) and (40) - (42) it is clear that $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$ are not arbitrary variables, just like $\{\delta\mathbf{v}, \delta\rho, \delta s, \delta\mathbf{B}\}$ are not arbitrary since their dynamics are constrained by the linearized ideal MHD equations.

In summary, Eqs. (6) - (9) and (39) - (42) combined are the complete set of constraints on ideal MHD perturbations. The algebraic constraints (6) - (8) and (39) express $\{\delta\mathbf{v}, \delta\rho, \delta s, \delta\mathbf{B}\}$ in terms of the new variables $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$, while the dynamical constraints (40) - (42) and (9) describe the evolution of the new variables $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$. It is straightforward to verify that the eight equations combined are equivalent to the linearized ideal

MHD equations. It is worthwhile to address that the eight constraints work together as a group. More specifically, the perturbation $\delta\mathbf{v}$ must satisfy both constraints (9) and (39) at the same time, for instance. In Ref. 15, Eq. (39) and Eq. (9) are claimed to be equivalent constraints on $\delta\mathbf{v}$. But as discussed above, they are not equivalent, but they coexist.

In the next section, these constraints will be used to calculate the second variation of the Hamiltonian and develop an energy principle.

V. THE ENERGY PRINCIPLE

As Eq. (37) suggests, with the variational constraints (39) and (6) - (8), the first variation of the Hamiltonian[2]

$$\delta H = \int \{(\rho\mathbf{v}\cdot\nabla s)\,\alpha + [\nabla\cdot(\rho\mathbf{v})]\beta - [\nabla\times(\mathbf{v}\times\mathbf{B})]\cdot\boldsymbol{\zeta} \\ + [\rho\mathbf{v}\nabla\cdot\mathbf{v} - (\nabla\times\mathbf{B})\times\mathbf{B} + \nabla p]\cdot\boldsymbol{\xi}\}\,\mathrm{d}^3x \tag{43}$$

vanishes when the system is at equilibrium, where the LHS of Eqs. (10) - (13) all equal zero. Then, using Eqs. (39) and (6) - (8) again, one further obtains the second variation of the Hamiltonian[2]

$$\delta^2 H = \int \rho(\delta\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla\mathbf{v})^2\,\mathrm{d}^3x + \delta^2 W, \tag{44}$$

where $\delta^2 W$ is the effective potential energy,

$$\delta^2 W = \int [\rho(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) - \rho(\mathbf{v}\cdot\nabla\boldsymbol{\xi})^2]\,\mathrm{d}^3x + \delta^2 W_0, \tag{45}$$

where the first two flow dependent terms come from the perturbed kinetic energy, while $\delta^2 W_0$ is the second variation of the potential energy,

$$\delta^2 W_0 = \int [\rho\partial_\rho p(\nabla\cdot\boldsymbol{\xi})^2 + (\nabla\cdot\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\nabla p) + \delta\mathbf{B}\cdot(\nabla\times\mathbf{B})\times\boldsymbol{\xi} + (\delta\mathbf{B})^2]\,\mathrm{d}^3x. \tag{46}$$

Note that so far we have only used the algebraic constraints (39) and (6) - (8), therefore $\delta\mathbf{v}$ in Eq. (44) has the form as in Eq. (39). However, as discussed in Sec. IV, $\delta\mathbf{v}$ must satisfy the dynamical constraint (9) at the same time. Substituting Eq. (9) into Eq. (44), one gets

$$\delta^2 H = \int \rho\dot{\boldsymbol{\xi}}^2\,\mathrm{d}^3x + \delta^2 W. \tag{47}$$

With the MHD Eqs. (10) - (13) the total Hamiltonian H can be proven to be conserved. For an equilibrium perturbed by a small perturbation $\boldsymbol{\xi}$, the Hamiltonian can be expanded as

$H = H_0 + \delta^2 H/2 + O[\boldsymbol{\xi}^3]$, where H_0 is the fixed background Hamiltonian, and $\delta^2 H \sim O[\boldsymbol{\xi}^2]$. Therefore $\delta^2 H$ is conserved as a result of the conservation of H. It has been proven in Ref. 4 and 16 that when $\delta^2 H$ has the form as in Eq. 47, the necessary and sufficient condition for the equilibrium to be stable is for $\delta^2 W$ to be positive for arbitrary $\boldsymbol{\xi}$. This is our energy principle for ideal MHD equilibria with flows. When there is no equilibrium flow ($\mathbf{v} = 0$), the first two terms in $\delta^2 W$ vanishes, and $\delta^2 W = \delta^2 W_0$ recovers the standard energy principle for equilibria with no flows[3]. However, note that the argument for the conservation of $\delta^2 H$ holds only when $\boldsymbol{\xi}$ is a small-amplitude linear perturbation, and therefore the criterion is only for the linear stability of the equilibrium.

As summarized in Sec. I, another stability criterion was previously obtained by Frieman and Rotenberg[1] from the linearized equation of motion, that the system is stable when $-\int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi})\, \mathrm{d}^3x$ is positive definite. One can verify that $-\int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi})\, \mathrm{d}^3x = \delta^2 W$, so such a stability criterion is equivalent to ours. However, the criterion was not explained in an energy perspective in Ref. 1. To turn it into an energy principle for further comparison with ours, we look into the effective Lagrangian and Hamiltonian constructed from the linearized equation of motion:

$$L_2 = \int [\rho \dot{\boldsymbol{\xi}}^2 + 2\rho \dot{\boldsymbol{\xi}} \cdot (\mathbf{v} \cdot \nabla \boldsymbol{\xi}) + \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi})]\, \mathrm{d}^3x, \tag{48}$$

$$H_2 = \int [\rho \dot{\boldsymbol{\xi}}^2 - \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi})]\, \mathrm{d}^3x. \tag{49}$$

They were included in the appendix of Ref. 1 as an entertainment without further study. However, by taking the time derivative of H_2 and using Eq. (1) and the self-adjointness of $\mathbf{F}(\boldsymbol{\xi})$, one finds that H_2 is conserved, so the system is stable when $-\int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi})\, \mathrm{d}^3x$ is positive definite[4]. As H_2 can be interpreted as the perturbed energy of the system, the criterion has become an energy principle. It is obvious that $H_2 = \delta^2 H$, so the criterion by Frieman and Rotenberg is indeed equivalent to ours discussed above. However, as H_2 is constructed from the linearized equation of motion, one must use the self-adjointness of $\mathbf{F}(\boldsymbol{\xi})$, the proof of which requires tedious algebra, to show that H_2 is conserved . On the other hand, obtained as the second variation of the total Hamiltonian, $\delta^2 H$ is intrinsically conserved, which avoids the algebra proving the self-adjointness of $\mathbf{F}(\boldsymbol{\xi})$. In fact, the self-adjointness of $\mathbf{F}(\boldsymbol{\xi})$ can be indirectly proven[5] as a result of the conservation of $\delta^2 H$.

In addition, the authors would like to address that in Ref. 1 the linearized equation of motion Eq. (1) was obtained with the equilibrium conditions and the constraints (6) -

(9). Furthermore, the most straightforward way to obtain Eqs. (48) and (49) is to use the equilibrium conditions and the constraints (6) - (9) to calculate the second variation of the action $\delta^2 S = \int L_2 \, \mathrm{d}t$, which directly gives the effective Lagrangian L_2, and H_2 and Eq. (1) can also be derived thereafter. But again, the conservation of H_2 has to be proven using the self-adjointness of $\mathbf{F}(\boldsymbol{\xi})$.

VI. DISCUSSION

As discussed in Sec. IV, Eqs. (39) and (6) - (8) were previously obtained by Morrison[10] and referred to as the "dynamically accessible" variations. Using them as constraints, Hameiri[2] obtained $\delta^2 H$ as in Eq. (44) where $\delta\mathbf{v}$ is in the form of Eq. (39) expressed in terms of $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$. The positivity of $\delta^2 H$ in such form with respect to arbitrary $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$ was then claimed by Hameiri to be the stability criterion of the equilibrium, which is more useful than the definiteness of $\delta^2 W$, because the arbitrariness of $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$ can be manipulated for stability analysis. Following are the reasons why we believe such a result is not valid.

To begin with, the "dynamically accessible" variations, (39) and (6) - (8), were obtained by Morrison[10] with the non-canonical Poisson bracket for the ideal MHD, which is not surprising because the Poisson bracket can also be derived from the Euler-Poincare reduction[8]. When these constraints were obtained, no derivation was shown other than an argument that they will automatically conserve all the Casimirs of the system. However, the existence of relevant Casimirs is required for such an argument to stand. Admittedly, Eqs. (6) - (8) correspond to the local conservation (advection) of mass, entropy, and magnetic flux respectively, but so far there has been no simple explicit Casimir found that corresponds to Eq. (39), despite the effort devoted[17,18]. On the other hand, in Sec. IV, Eq. (39) is, for the first time, rigorously derived without any new Casimirs concerned. Therefore we believe Eq. (39) is just a result of the Euler-Poincare structure of the ideal MHD, as shown in our derivation, and the fundamental constants of motion in the ideal MHD are just the advected quantities: the mass, the entropy, and the magnetic flux.

Moreover, in Ref. 2, $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$ are claimed to be arbitrary variables, which can be used for stability analysis. However, by considering the dynamics of the perturbations, we have shown that $\{\boldsymbol{\xi}, \beta, \alpha, \boldsymbol{\zeta}\}$ are in fact not arbitrary as their dynamics are constrained by Eqs. (9) and (40) - (42), so such a claim does not stand. It was also claimed that (44) is a better expression of $\delta^2 H$ than (47), but given the fact that $\delta\mathbf{v}$ must satisfy constraints (9) and (39) simultaneously, Eqs. (44) and (47) are equivalent expressions for $\delta^2 H$. In fact, we find the latter expression of $\delta^2 H$ more useful, since it has been proven when $\delta^2 H$ is conserved and in the form of Eq. (47), the definiteness of $\delta^2 W$ determines the stability of the equilibria[4].

Last but not least, in Refs. 2, 9, and 10, the reason why the definiteness of $\delta^2 H$ can determine the stability of the equilibrium was not clearly explained. As far as we can understand from their discussion, it is a noncanonical analog of the Dirichlet's theorem[10,12], which has only been proven for canonical systems. The core of such an analog is in fact the energy Casimir method[11,12], based on their interpretation that the dynamically accessible variations are equivalent to the Casimirs. But as discussed above, such an interpretation may not be correct. We believe that the role of $\delta^2 H$ in the stability concern is a conserved quantity, which is a dynamical effect and was not discussed in their argument. In addition, it is implied in Ref. 2, 9, and 10 that the definiteness of $\delta^2 H$ is a criterion for the nonlinear stability of the ideal MHD equilibria, while our energy principle is for the linear stability.

In this paper we have proven the necessary and sufficient condition for ideal MHD equilibria with flows to be linearly stable is for $\delta^2 W$ in Eq. (45) as the effective potential energy to be positive for arbitrary $\boldsymbol{\xi}$. The criterion is found equivalent to what Frieman and Rotenberg previously proposed, but derived more straightforwardly from the second variation of the Hamiltonian calculated with proper constraints. The difference from the standard energy principle for ideal MHD equilibria with no flows are the first two flow-dependent terms which come from the perturbed kinetic energy. In principle, any flow driven ideal MHD instability, should come from these two terms. For example, the magnetorotational instability (MRI) has been discussed from such a perspective[19].

The derivation for the complete set of variational constraints applies to not only the ideal MHD, but any continuum mechanics that has the Euler-Poincare structure. Therefore such an approach for linear stability analysis can easily be applied to other systems, say, the Vlasov-Maxwell system.

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