Princeton Plasma Physics Laboratory

PPPL- 5023

PPPL- 5023

Field Theory And Weak Euler-Langrange Equation For Classical Particle-field Systems

Hong Qin, Joshua W. Burby and Ronald C. Davidson

APRIL, 2014

Prepared for the U.S. Department of Energy under Contract DE-AC02-09CH11466.

Princeton Plasma Physics Laboratory
Report Disclaimers

Field theory and weak Euler-Langrange equation for classical particle-field systems

Hong Qin,[1,2] Joshua W. Burby,[2] and Ronald C. Davidson[2]

[1]*Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

[2]*Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543*

Abstract

It is commonly believed that energy-momentum conservation is the result of space-time symmetry. However, for classical particle-field systems, e.g., Klimontovich-Maxwell and Klimontovich-Poisson systems, such a connection hasn't been formally established. The difficulty is due to the fact that particles and the electromagnetic fields reside on different manifolds. To establish the connection, the standard Euler-Lagrange equation needs to be generalized to a weak form. Using this technique, energy-momentum conservation laws that are difficult to find otherwise can be systematically derived.

PACS numbers: 52.25.Dg, 03.50.Kk

The classical non-relativistic particle-field system in flat space is governed by the Newton-Maxwell equations

$$\ddot{\boldsymbol{X}}_{sp} = \left(\frac{q}{m}\right)_s \left(\boldsymbol{E} + \frac{1}{c}\dot{\boldsymbol{X}}_{sp} \times \boldsymbol{B}\right), \tag{1}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi \sum_{s,p} q_s \delta(\boldsymbol{X}_{sp} - \boldsymbol{x}),\ \ \nabla \times \boldsymbol{B} = \frac{4\pi}{c} \sum_{s,p} q_s \dot{\boldsymbol{X}}_{sp} \delta(\boldsymbol{X}_{sp} - \boldsymbol{x}) + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t},\ \ \ \nabla \cdot \boldsymbol{B} = 0, \tag{3}$$

where $\boldsymbol{X}_{sp}$ is the trajectory of the p-th particle of the s-species, and q_s and m_s are the particle charge and mass, respectively. Equations (1)-(2) can be expressed equivalently in the form of Klimontovich-Maxwell (KM) equations [1]

$$\frac{\partial F_s}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F_s}{\partial \boldsymbol{x}} + \left(\frac{q}{m}\right)_s \left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \frac{\partial F_s}{\partial \boldsymbol{v}} = 0, \tag{4}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi \sum_s q_s \int F_s d_s^3 \boldsymbol{v},\ \ \nabla \times \boldsymbol{B} = \frac{4\pi}{c} \sum_s q_s \int F_s \boldsymbol{v} d^3\boldsymbol{v} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{5}$$

where $F_s(\boldsymbol{x}, \boldsymbol{v}, t) = \sum_p \delta(\boldsymbol{X}_{sp} - \boldsymbol{x}) \delta(\dot{\boldsymbol{X}}_{sp} - \boldsymbol{v})$ is the Klimontovich distribution function in the phase space $(\boldsymbol{x}, \boldsymbol{v})$.

Reduced models are often used in plasma physics. For example, the electrostatic Klimontovich-Poisson (KP) system is given by

$$\frac{\partial F_s}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F_s}{\partial \boldsymbol{x}} + \left(\frac{q}{m}\right)_s \left(-\nabla\phi + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}_0\right) \cdot \frac{\partial F_s}{\partial \boldsymbol{v}} = 0, \tag{6}$$

$$\nabla^2 \phi = -4\pi \sum_s q_s \int F_s d^3\boldsymbol{v}, \tag{7}$$

where $\boldsymbol{B}_0(\boldsymbol{x})$ is a background magnetic field produced by steady external currents, and $\boldsymbol{E} = -\nabla\phi$ is the longitudinal electric field. Another well-known model is the Klimontovich-Darwin (KD) system [2–5],

$$\frac{\partial F_s}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F_s}{\partial \boldsymbol{x}} + \left(\frac{q}{m}\right)_s \left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \frac{\partial F_s}{\partial \boldsymbol{v}} = 0, \tag{8}$$

$$\nabla^2\phi + \nabla \cdot \left(\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right) = -4\pi \sum_s q_s \int F_s d^3\boldsymbol{v},\ \ \nabla \times (\nabla \times \boldsymbol{A}) + \frac{1}{c}\frac{\partial \nabla\phi}{\partial t} = \frac{4\pi}{c} \sum_s q_s \int F_s \boldsymbol{v} d^3\boldsymbol{v}, \tag{9}$$

where $\boldsymbol{E} \equiv -(1/c)\partial \boldsymbol{A}/\partial t - \nabla\phi$ and $\boldsymbol{B} \equiv \nabla \times \boldsymbol{A}$ are the electric and magnetic fields.

The local energy-momentum conservation laws for the Klimontovich-Maxwell system (4)-

(5) is well-known [6],

$$\frac{\partial}{\partial t}\left[\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}+\sum_{s,p}\frac{m_s\dot{\boldsymbol{X}}_{sp}^2}{2}\delta_2\right]+\nabla\cdot\left[\frac{c\boldsymbol{E}\times\boldsymbol{B}}{4\pi}+\sum_{s,p}\frac{m_s\dot{\boldsymbol{X}}_{sp}^2}{2}\dot{\boldsymbol{X}}_{sp}\delta_2\right]=0, \tag{10}$$

$$\frac{\partial}{\partial t}\left[\frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi c}+\sum_{s,p}m_s\dot{\boldsymbol{X}}_{sp}\delta_2\right]+\nabla\cdot\left[\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{8\pi}I-\frac{\boldsymbol{E}\boldsymbol{E}+\boldsymbol{B}\boldsymbol{B}}{4\pi}+\sum_{s,p}m_s\dot{\boldsymbol{X}}_{sp}\dot{\boldsymbol{X}}_{sp}\delta_2\right]=0, \tag{11}$$

where we have introduced $\delta_2\equiv\delta(\boldsymbol{X}_{sp}-\boldsymbol{x})$ to simplify the notation. These conservation laws can be expressed equivalently in terms of the distribution function F_s through the following identities,

$$\sum_p\frac{\dot{\boldsymbol{X}}_{sp}^2}{2}\delta_2=\int d^3\boldsymbol{v}F_s\frac{\boldsymbol{v}^2}{2},\;\;\sum_p\frac{\dot{\boldsymbol{X}}_{sp}^2}{2}\dot{\boldsymbol{X}}_{sp}\delta_2=\int d^3\boldsymbol{v}F_s\frac{\boldsymbol{v}^2}{2}\boldsymbol{v}, \tag{12}$$

$$\sum_p\dot{\boldsymbol{X}}_{sp}\delta_2=\int d^3\boldsymbol{v}F_s\boldsymbol{v},\;\;\sum_p\dot{\boldsymbol{X}}_{sp}\dot{\boldsymbol{X}}_{sp}\delta_2=\int d^3\boldsymbol{v}F_s\boldsymbol{v}\boldsymbol{v}. \tag{13}$$

For the reduced systems, e.g., the KP system and the KD system, it is also critical to know the exact local energy-momentum conservation laws admitted by the models. In practical applications, such as current drive with lower-hybrid waves, and electrostatic drift wave turbulence, such local energy-momement conservation laws for the reduced system have profound implications [7–10] . We emphasize that we are looking for the exact conservation laws admitted by the KP and KD systems, which are not exact special cases of the KM system, and should be viewed as independent systems in their own right. For example, we cannot take the exact energy-momentum equations (10) and (11), and approximate $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$ to obtain the exact energy-momentum conservation law for the KP system, even though the conservation law obtained this way could be an approximate one for the KP system. The existence of exact local conservation laws is a necessary condition for the models to be theoretically well-posed and for the validity of particle simulations based on the KP or KD systems [4]. However, exact local energy-momentum conservation laws for the reduced systems are not easy to find, and systematic methods to derive these local conservation laws are not yet available.

On the other hand, conservation laws and symmetries are closely related. It is commonly believed that, according to Noether's theorem [11, 12], conservation laws can be derived from the symmetries of the corresponding field theories. In standard field theories, this is certainly true, and the symmetry in time for the action is related to energy conservation, and the symmetry in space corresponds to momentum conservation. Therefore, it is reasonable to expect that by analyzing the symmetries of the actions and Lagrangian densities for our reduced systems, we may be able to systematically derive the desired conservation laws. However, it is surprising to find out that for particle-field systems considered here, the field theory works differently. First, let's recall the action and Lagrangian density for the KM system given by Low [13],

$$\mathcal{A}[\phi, \boldsymbol{A}, \boldsymbol{X}_{sp}] = \int L_{KM} d^3\boldsymbol{x}dt,\ L_{KM} = L_{KMF} + L_{KMP}, \tag{14}$$

$$L_{KMF} = \left(\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi\right)^2 /8\pi - \left(\nabla \times \boldsymbol{A}\right)^2 /8\pi, \tag{15}$$

$$L_{KMP} = \sum_{s,p} \left[-q_s\phi + \frac{q_s}{c}\dot{\boldsymbol{X}}_{sp} \cdot \boldsymbol{A} + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right] \delta_2. \tag{16}$$

It is straightforward to verify that Eqs. (1) -(3) follow from $\delta\mathcal{A}/\delta\boldsymbol{X}_{sp} = 0$, $\delta\mathcal{A}/\delta\phi = 0$, and $\delta\mathcal{A}/\delta\boldsymbol{A} = 0$. For the KP system, the action and Lagrangian density are given by

$$\mathcal{A}[\phi, \boldsymbol{X}_{sp}] = \int L_{KP} d^3\boldsymbol{x}dt,\ L_{KP} = L_{KPF} + L_{KPP}, \tag{17}$$

$$L_{KPF} = (\nabla\phi)^2/8\pi,\ \ L_{KPP} = \sum_{s,p} \left[-q_s\phi + \frac{q_s}{c}\dot{\boldsymbol{X}}_{sp} \cdot \boldsymbol{A}_0 + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right] \delta_2, \tag{18}$$

where $\boldsymbol{A}_0$ is the vector potential for a given external magnetic field $\boldsymbol{B}_0 = \nabla \times \boldsymbol{A}_0$. For the KD system, the action and Lagrangian density for the KM system are

$$\mathcal{A}[\phi, \boldsymbol{A}, \boldsymbol{X}_{sp}] = \int L_{KD} d^3\boldsymbol{x}dt,\ L_{KD} = L_{KDF} + L_{KDP}, \tag{19}$$

$$L_{KDF} = \left[\frac{2}{c}\nabla\phi \cdot \frac{\partial \boldsymbol{A}}{\partial t} + (\nabla\phi)^2\right] /8\pi - \left(\nabla \times \boldsymbol{A}\right)^2 /8\pi, \tag{20}$$

$$L_{KDP} = \sum_{p=1}^{N} \left[-q_s\phi + \frac{q_s}{c}\dot{\boldsymbol{X}}_{sp} \cdot \boldsymbol{A} + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right] \delta_2. \tag{21}$$

Based on the spirit of Noether's theorem, we would like to determine whether the local energy-momentum conservation laws can be derived from the symmetries of the corresponding Lagrangian density. It turns out that the answer to this question is not as simple as that in standard field theory. This is because the fields in the present field theory, i.e., $\boldsymbol{X}_{sp}(t)$, $\phi(\boldsymbol{x}, t)$, and $\boldsymbol{A}(\boldsymbol{x}, t)$ are defined on different domains. The potentials are defined on the space-time domain $(\boldsymbol{x}, t)$, whereas the particle trajectory $\boldsymbol{X}_{sp}(t)$ is only defined on the time-axis. This unique feature has not been discussed before, and it makes a significant difference in the formulation of the field theory presented here.

In this Letter, we develop the field theory for classical particle-systems with this feature, in particular, the KM system, the KP system, and the KD system. The most distinct characteristic of the field theory presented here is that the field equation for $\boldsymbol{X}_{sp}(t)$ assumes a form we call the weak Euler-Lagrange (EL) equation, which is different from the standard Euler-Lagrange equation. The necessity of using this weak EL equation is mandated by the fact that $\boldsymbol{X}_{sp}(t)$, as a field, is not defined on the entire space-time domain, but only on the time-axis. The weak EL equation with respect to $\boldsymbol{X}_{sp}(t)$ plays an indispensable role in the symmetry analysis and derivation of local conservation laws. For the KP system and KD system, the analysis developed here enables us to find the desired local conservation laws, which have not been systematically discussed in the literature. For the KM system, where the local energy-momentum conservation laws (10)-(11) are known, our analysis serves the purpose of establishing a connection between the energy-momentum conservation laws and symmetries of the Lagrangian density L_{KM}. Interestingly, such a connection has only been cautiously suggested [6] but not explicitly established previously. It is not surprising though, because the weak EL equation developed here is needed to establish the connection. Due to the space limitation, we will present the detailed derivation of the field theory and new conservation laws only for the KP system of a magnetized plasma, and list the main results for the KM system and KD system at the end of this paper.

In plasma physics, one often works with the Vlasov-Maxwell (VM) system, Equations (4)-(5) recover the VM equations when two-particle correlations (collisions) become negligibly small as the number of particles becomes increasingly large, while holding total charge and and charge to mass ratio fixed. In the present study, we work with the Klimontovich-Maxwell system (4)-(5) or (1)-(2) and pass to the limit of the Vlasov-Maxwell system when necessary under the assumption of negligible collisions. Similarly, the Vlasov-Poisson (VP) and Vlasov-Darwin (VD) systems are regarded as the collisionless limits of the KP and KD systems, respectively. We also note that while our focus here is on the particle-field system, Eulerian field theories for the VM and VP systems have been developed by Morrison et al. [14–17] using a variety of theoretical constructions. In Eulerian theories, the particle distribution in phase space replaces $\boldsymbol{X}_{sp}(t)$ as the field variable.

We begin with Eq. (17) for the KP system, and determine how the action and Lagrangian density vary in response to the field variation $\delta \boldsymbol{X}_{sp}$ and $\delta\phi(\boldsymbol{x},t)$,

$$\delta\mathcal{A} = \int d^3\boldsymbol{x}dt\,\delta\phi E_\phi(L_{KP}) + \sum_{s,p=1}^{N}\int dt\,\delta\boldsymbol{X}_{sp}\cdot\int d^3\boldsymbol{x}\,E_{\boldsymbol{X}_{sp}}(L_{KP}), \tag{22}$$

$$E_\phi(L_{KP}) \equiv \frac{\partial L_{KP}}{\partial\phi} - \frac{D}{Dx^i}\frac{\partial L_{KP}}{\partial\phi_{,i}},\ \ E_{\boldsymbol{X}_{sp}}(L_{KP}) \equiv \frac{\partial L_{KP}}{\partial\boldsymbol{X}_{sp}} - \frac{D}{Dt}\left(\frac{\partial L_{KP}}{\partial\dot{\boldsymbol{X}}_{sp}}\right). \tag{23}$$

In Eq. (22), $\phi_{,i} \equiv \partial\phi/\partial x^i$ and integration by parts has been applied with respect to terms containing $\partial L_{KP}/\partial\phi_{,i}$ and $\partial L_{KP}/\partial\dot{\boldsymbol{X}}_{sp}$. Here, $E_\phi(L_{KP})$ and $E_{\boldsymbol{X}_{sp}}(L_{KP})$ are the Euler operators with respect to ϕ and $\boldsymbol{X}_{sp}$, respectively. For a variable h, Dh/Dx^i and Dh/Dt represent the space-time derivatives when $h = h(\boldsymbol{x},t)$ is considered as a field on the space-time domain. Because $\delta\phi(\boldsymbol{x},t)$ is arbitrary, $\delta\mathcal{A}/\delta\phi = 0$ requires the EL equation for ϕ to hold, i.e., $E_\phi(L_{KP}) = 0$, which is indeed the Poisson equation (7), as expected. The field equation for $\boldsymbol{X}_{sp}$ is more interesting. Because $\delta\boldsymbol{X}_{sp}$ is arbitrary only on the time-axis, the condition $\delta\mathcal{A}/\delta\boldsymbol{X}_{sp} = 0$ requires only that the integral of $E_{\boldsymbol{X}_{sp}}(L_{KP})$ over space vanish, i.e.,

$$\int d^3\boldsymbol{x} E_{\boldsymbol{X}_{sp}}(L_{KP}) = 0. \tag{24}$$

Equation (24) will be called the submanifold Euler-Lagrangian equation because it is defined only on the time-axis after the integrating over the spatial variable. If $\boldsymbol{X}_{sp}$ were a function of the entire space-time domain, then $E_{\boldsymbol{X}_{sp}}(L_{KP})$ would vanish everywhere, as in the case for $\phi(\boldsymbol{x},t)$. In general, we expect that $E_{\boldsymbol{X}_{sp}}(L_{KP}) \neq 0$.

We now derive an explicit expression for $E_{\boldsymbol{X}_{sp}}(L_{KP})$. For the first term in $E_{\boldsymbol{X}_{sp}}(L_{KP})$,

$$\begin{aligned}\frac{\partial L_{KP}}{\partial\boldsymbol{X}_{sp}} &= \left[\frac{q_s}{c}\boldsymbol{A}_0\cdot\dot{\boldsymbol{X}}_{sp} - q_s\phi + \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right]\frac{\partial\delta_2}{\partial\boldsymbol{X}_{sp}} \\ &= \frac{\partial}{\partial\boldsymbol{x}}\left[\left(q_s\phi - \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2 - \frac{q_s}{c}\boldsymbol{A}_0\cdot\dot{\boldsymbol{X}}_{sp}\right)\delta_2\right] + \left[\frac{q_s}{c}\frac{\partial\boldsymbol{A}_0}{\partial\boldsymbol{x}}\cdot\dot{\boldsymbol{X}}_{sp} - q_s\frac{\partial\phi}{\partial\boldsymbol{x}}\right]\delta_2.\end{aligned} \tag{25}$$

The second term in $E_{\boldsymbol{X}_{sp}}(L_{KP})$ is given by

$$\begin{aligned}\frac{D}{Dt}\frac{\partial L_{KP}}{\partial\dot{\boldsymbol{X}}_{sp}} &= m_s\ddot{\boldsymbol{X}}_{sp}\delta_2 + \left(m_s\dot{\boldsymbol{X}}_{sp} + \frac{q_s}{c}\boldsymbol{A}_0\right)\frac{\partial\delta_2}{\partial t} \\ &= m_s\ddot{\boldsymbol{X}}_{sp}\delta_2 - \frac{\partial}{\partial\boldsymbol{x}}\cdot\left[\dot{\boldsymbol{X}}_{sp}(m_s\dot{\boldsymbol{X}}_{sp} + \frac{q_s}{c}\boldsymbol{A}_0)\delta_2\right] + \frac{q_s}{c}\dot{\boldsymbol{X}}_{sp}\cdot\frac{\partial\boldsymbol{A}_0}{\partial\boldsymbol{x}}\delta_2.\end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}E_{\boldsymbol{X}_{sp}}(L_{KP}) = &\left[\frac{q_s}{c}\left(\frac{\partial\boldsymbol{A}_0}{\partial\boldsymbol{x}}\cdot\dot{\boldsymbol{X}}_{sp} - \dot{\boldsymbol{X}}_{sp}\cdot\frac{\partial\boldsymbol{A}_0}{\partial\boldsymbol{x}}\right) - q_s\frac{\partial\phi}{\partial\boldsymbol{x}} - m_s\ddot{\boldsymbol{X}}_{sp}\right]\delta_2 \\ &+ \frac{\partial}{\partial\boldsymbol{x}}\left[\left(-\frac{q_s}{c}\boldsymbol{A}_0\cdot\dot{\boldsymbol{X}}_{sp} + q_s\phi - \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right] + \frac{\partial}{\partial\boldsymbol{x}}\cdot\left[\dot{\boldsymbol{X}}_{sp}\left(m_s\dot{\boldsymbol{X}}_{sp} + \frac{q_s}{c}\boldsymbol{A}_0\right)\delta_2\right].\end{aligned} \tag{27}$$

Substituting Eq. (27) into the submanifold EL equation (24), we immediately recover Newton's equation for $\boldsymbol{X}_{sp}$, i.e., $m_s\ddot{\boldsymbol{X}}_{sp}/q_s = -\partial\phi/\partial\boldsymbol{x} + \dot{\boldsymbol{X}}_{sp}\times\boldsymbol{B}_0/c$, which reduces Eq. (27) to

$$\begin{aligned} E_{\boldsymbol{X}_{sp}}(L_{KP}) &\equiv \frac{\partial L_{KP}}{\partial \boldsymbol{X}_{sp}} - \frac{D}{Dt}\left(\frac{\partial L_{KP}}{\partial \dot{\boldsymbol{X}}_{sp}}\right) \\ &= \frac{\partial}{\partial\boldsymbol{x}}\left[\left(-\frac{q_s}{c}\boldsymbol{A}_0\cdot\dot{\boldsymbol{X}}_{sp} + q_s\phi - \frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right] + \frac{\partial}{\partial\boldsymbol{x}}\cdot\left[\dot{\boldsymbol{X}}_{sp}\left(m_s\dot{\boldsymbol{X}}_{sp} + \frac{q_s}{c}\boldsymbol{A}_0\right)\delta_2\right]. \end{aligned} \tag{28}$$

As expected, $E_{\boldsymbol{X}_{sp}}(L_{KP}) \neq 0$. We will refer to Eq. (28) as the weak Euler-Lagrange equation, which is the foundation for the subsequent analysis of the local conservation laws. The qualifier "weak" is used to indicate the fact that only the spatial integral of the Euler derivative $E_{\boldsymbol{X}_{sp}}(L_{KP})$ is zero [see Eq. (24)], in comparison with the standard EL equation, which demands that the Euler derivative vanishes everywhere.

We define a symmetry of the action $\mathcal{A}[\phi,\boldsymbol{X}_{sp}]$ to be a group of transformation $(\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp}) \mapsto (\tilde{\boldsymbol{x}},\tilde{t},\tilde{\phi},\tilde{\boldsymbol{X}}_{sp})$ such that $\int L_{KP}[\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp}]d^3\boldsymbol{x}dt = \int L_{KP}[\tilde{\boldsymbol{x}},\tilde{t},\tilde{\phi},\tilde{\boldsymbol{X}}_{sp}]d^3\tilde{\boldsymbol{x}}d\tilde{t}$. If the symmetry is generated by a vector field on the space of $(\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp})$, $V = \boldsymbol{\xi}\cdot\partial/\partial\boldsymbol{x} + \xi^t\partial/\partial t + \psi\partial/\partial\phi + \boldsymbol{Y}_p\cdot\partial/\partial\boldsymbol{X}_{sp}$, then the infinitesimal criteria of invariance is given by [12]

$$prV(L) + L\,Div\xi = 0, \tag{29}$$

where prV is the prolongation of the vector field V on $(\boldsymbol{x},t,\phi,\boldsymbol{X}_{sp})$, and $Div\xi$ is the divergence of the vector field $\xi = \boldsymbol{\xi}\cdot\partial/\partial\boldsymbol{x} + \xi^t\partial/\partial t$ on the space-time domain. Given the symmetry vector field V, the infinitesimal criteria for invariance will generate the desired conservation law corresponding to the symmetry vector field V, after use is made of the EL equation as well as the weak EL equation for the systems in the present study. We first look for the symmetry group that generates local energy conservation. The group of transformation $(\tilde{\boldsymbol{x}},\tilde{t},\tilde{\phi},\tilde{\boldsymbol{X}}_{sp}) = (\boldsymbol{x},t+\epsilon,\phi,\boldsymbol{X}_{sp})$ for $\epsilon\in R$ is a symmetry of L_{KP}, because L_{KP} does not depend on t explicitly, i.e., $\partial L_{KP}/\partial t = 0$, which can be written as

$$\frac{DL_{KP}}{Dt} - \phi_{,t}\frac{\partial L_{KP}}{\partial\phi} - \phi_{,jt}\frac{\partial L_{KP}}{\partial\phi_{,j}} - \sum_{s,p}\left(\dot{\boldsymbol{X}}_{sp}\cdot\frac{\partial L_{KP}}{\partial\boldsymbol{X}_{sp}} + \ddot{\boldsymbol{X}}_{sp}\cdot\frac{\partial L_{KP}}{\partial\dot{\boldsymbol{X}}_{sp}}\right) = 0. \tag{30}$$

Equation (30) is the special form of Eq. (29) for this symmetry group. From the EL equation for ϕ, i.e., $E_\phi(L_{KP}) = 0$, we obtain

$$\phi_{,t}\frac{\partial L_{KP}}{\partial\phi} + \phi_{,jt}\frac{\partial L_{KP}}{\partial\phi_{,j}} = \frac{D}{Dx^j}\left(\phi_{,t}\frac{\partial L_{KP}}{\partial\phi_{,j}}\right). \tag{31}$$

The weak EL equation for $\boldsymbol{X}_{sp}$, i.e., Eq. (28), gives

$$\dot{\boldsymbol{X}}_{sp}\frac{\partial L_{KP}}{\partial \boldsymbol{X}_{sp}}+\ddot{\boldsymbol{X}}_{sp}\frac{\partial L_{KP}}{\partial \dot{\boldsymbol{X}}_{sp}}=\frac{\partial}{\partial \boldsymbol{x}}\cdot\left[\dot{\boldsymbol{X}}_{sp}\left(q_s\phi+\frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right]+\frac{\partial}{\partial t}\left(\dot{\boldsymbol{X}}_{sp}\frac{\partial L_{KP}}{\partial \dot{\boldsymbol{X}}_{sp}}\delta_2\right) \tag{32}$$

Combining Eqs. (31) and (32), we obtain the first local energy conservation law,

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{8\pi}-\sum_{s,p}\left(q_s\phi+\frac{m_s}{2}\dot{\boldsymbol{X}}^2{}_{sp}\right)\delta_2\right]+\frac{\partial}{\partial \boldsymbol{x}}\cdot\left[\frac{-1}{4\pi}\phi_{,t}\nabla\phi-\sum_{s,p}\dot{\boldsymbol{X}}_{sp}\left(q_s\phi+\frac{m_s}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right]=0. \tag{33}$$

We subtract the identify

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{4\pi}+\phi\nabla^2\phi\right]+\frac{1}{4\pi}\frac{\partial}{\partial \boldsymbol{x}}\cdot\left(-\phi_{,t}\nabla\phi-\phi\nabla\phi_{,t}\right)=0 \tag{34}$$

from Eq. (33) to express the energy conservation law in another equivalent form

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{8\pi}+\sum_{s,p}\frac{m_s\dot{\boldsymbol{X}}_{sp}^2}{2}\delta_2\right]+\frac{\partial}{\partial \boldsymbol{x}}\cdot\left[\sum_{s,p}\dot{\boldsymbol{X}}_{sp}\left(q_s\phi+\frac{m_s\dot{\boldsymbol{X}}_{sp}^2}{2}\right)\delta_2-\frac{1}{4\pi}\phi\nabla\phi_{,t}\right]=0. \tag{35}$$

In terms of the distribution function F_s, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2}{8\pi}+\sum_s\int F_s\frac{m_s\boldsymbol{v}^2}{2}d^3\boldsymbol{v}\right]+\\&\frac{\partial}{\partial \boldsymbol{x}}\cdot\left(\sum_s\int F_s\frac{m_s\boldsymbol{v}^2}{2}\boldsymbol{v}d^3\boldsymbol{v}+\sum_s q_s\phi\int F_s\boldsymbol{v}d^3\boldsymbol{v}-\frac{1}{4\pi}\phi\nabla\phi_{,t}\right)=0.\end{aligned} \tag{36}$$

We emphasize again that Eq. (36) is the exact energy conservation law admitted by the KP system Eqs. (6) and (7), and it cannot be obtained by replacing $\boldsymbol{E}$ by $-\nabla\phi$ and $\boldsymbol{B}$ by $\boldsymbol{B}_0$ in the conservation law for the KM sytem (10) and (11). The sum of the last two terms in Eq. (36) is the electrostatic Ponyting flux of the KP system, first discussed by Similon [18] for an unmagnetized plasma by algebraic manipulation. Its importance for electrostatic particle simulations was addressed by Decyk [19]. Here, it appears naturally as a consequence of the symmetry analysis. We observe that the external $\boldsymbol{B}_0$ does not contribute to the energy flux of the electromagnetic field.

Our next goal is to search for the symmetry that generates the momentum conservation law. In standard field theories, if the Lagrangian density does not depend on $\boldsymbol{x}$ explicitly, then it admits the symmetry of spatial translation, $\tilde{\boldsymbol{x}}=\boldsymbol{x}+\epsilon\boldsymbol{u}$, for a constant vector $\boldsymbol{u}$ and $\epsilon\in R$. Then the usual form of Noether's theorem leads to momentum conservation. This strategy does not work here because L_{KP} depends on $\boldsymbol{x}$ explicitly through $\delta_2\equiv\delta(\boldsymbol{X}_{sp}-\boldsymbol{x})$

and $\boldsymbol{A}_0(\boldsymbol{x})$. However, if we simultaneously translate both $\boldsymbol{x}$ and $\boldsymbol{X}_{sp}$ by the same amount, then δ_2 is invariant. Thus, we consider the translational transformation

$$(\tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\phi}, \tilde{\boldsymbol{X}}_{sp}) = (\boldsymbol{x} + \epsilon \boldsymbol{u}, t, \phi, \boldsymbol{X}_{sp} + \epsilon \boldsymbol{u}), \tag{37}$$

under which $\tilde{\phi}(\tilde{\boldsymbol{x}}) = \phi(\boldsymbol{x}) = \phi(\tilde{\boldsymbol{x}} - \epsilon \boldsymbol{u})$ and $\tilde{\boldsymbol{X}}_{sp}(\tilde{t}) = \boldsymbol{X}_{sp}(t) + \epsilon \boldsymbol{u}$. When $\boldsymbol{A}_0(\boldsymbol{x}) = 0$, we can verify that Eq. (29) is satisfied, and Eq. (37) is indeed a symmtry admitted by L_{KP}. The corresponding vector field is $V = \partial/\partial \boldsymbol{x} + \sum_{sp} \partial/\partial \boldsymbol{X}_{sp}$, and $PrV = V$ since V is a constant. The notation $\partial/\partial \boldsymbol{x}$ here represents $\partial/\partial x^i$ for $i = 1, 2, 3$. In this case, the infinitesimal criteria of invariance in (29) is

$$\frac{\partial L_{KP}}{\partial \boldsymbol{x}} + \sum_p \frac{\partial L}{\partial \boldsymbol{X}_{sp}} = 0. \tag{38}$$

When $\boldsymbol{A}_0(\boldsymbol{x}) \neq 0$, the right-hand side of Eq. (38) will have a source term, and instead we obtain

$$\frac{\partial L_{KP}}{\partial \boldsymbol{x}} + \sum_{s,p} \frac{\partial L}{\partial \boldsymbol{X}_{sp}} = \sum_{s,p} \dot{\boldsymbol{X}}_{sp} \cdot \frac{\partial \boldsymbol{A}_0}{\partial \boldsymbol{x}} \delta_2. \tag{39}$$

It will be clear shortly that this term represents part of the momentum input due to the external magnetic field through the Lorentz force. For the first term in Eq. (39), we invoke the EL equation $E_\phi(L_{KP}) = 0$ to obtain

$$\frac{\partial L_{KP}}{\partial \boldsymbol{x}} = \frac{D L_{KP}}{D \boldsymbol{x}} - \frac{D}{D x^j} \left(\frac{\partial L_{KP}}{\partial \phi_{,j}} \nabla \phi \right). \tag{40}$$

For the second term in Eq. (39), the weak EL equation for $\boldsymbol{X}_{sp}$ (27) is applied, which gives

$$\begin{aligned} \frac{\partial L}{\partial \boldsymbol{X}_{sp}} =& \frac{d}{dt} \left[\left(m_s \dot{\boldsymbol{X}}_{sp} + \frac{q_s}{c} \boldsymbol{A}_0 \right) \delta_2 \right] + \frac{\partial}{\partial \boldsymbol{x}} \left[\left(\frac{q_s}{c} \boldsymbol{A}_0 \cdot \dot{\boldsymbol{X}}_{sp} + q_s \phi - \frac{m_s \dot{\boldsymbol{X}}_{sp}^2}{2} \right) \delta_2 \right] \\ &+ \frac{\partial}{\partial \boldsymbol{x}} \cdot \left[\dot{\boldsymbol{X}}_{sp} \left(m_s \dot{\boldsymbol{X}}_{sp} + \frac{q_s}{c} \boldsymbol{A}_0 \right) \delta_2 \right]. \end{aligned} \tag{41}$$

Therefore, conservation law generated by Eq. (39) is

$$\frac{\partial}{\partial t} \left(\sum_{s,p} m_s \dot{\boldsymbol{X}}_{sp} \delta_2 \right) + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left[\sum_{s,p} m_s \dot{\boldsymbol{X}}_{sp} \dot{\boldsymbol{X}}_{sp} \delta_2 + \frac{\boldsymbol{I}}{8\pi} (\nabla \phi)^2 - \frac{1}{4\pi} \nabla \phi \nabla \phi \right] = \sum_{s,p} m_s \frac{\dot{\boldsymbol{X}}_{sp}}{c} \times \boldsymbol{B}_0 \delta_2. \tag{42}$$

Evidently, this is the local conservation law of momentum. In term of the distribution function F_s, it can expressed as

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sum_s m_s \int F_s \boldsymbol{v} d^3 \boldsymbol{v} \right) + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left[\sum_s m_s \int F_s \boldsymbol{v} \boldsymbol{v} d^3 \boldsymbol{v} + \frac{\boldsymbol{I}}{8\pi} (\nabla \phi)^2 - \frac{1}{4\pi} \nabla \phi \nabla \phi \right] \\ = \sum_s q_s \left(\int F_s \frac{\boldsymbol{v}}{c} d^3 \boldsymbol{v} \right) \times \boldsymbol{B}_0. \end{aligned} \tag{43}$$

The first term on the left-hand side of Eq. (42) or Eq. (43) is the rate of variation of the momentum density, the second term is the divergence of the flux, and the term on the right-hand side is the momentum input due to the background magnetic field. Note that the momentum density is purely mechanical, and does not include the electromagnetic momentum density $-\nabla\phi \times \boldsymbol{B}_0/4\pi c$. This is not totally intuitive. This conservation law is the result of the symmetry (37), which is different from the well-known translational symmetry for standard field theory. Because L_{KP} depends on $\boldsymbol{x}$ explicitly through $\delta_2 \equiv \delta(\boldsymbol{X}_{sp} - \boldsymbol{x})$, a translation in $\boldsymbol{x}$ alone is not a symmetry of L_{KP}, even when $\boldsymbol{A}_0(\boldsymbol{x}) = 0$. Instead, the symmetry group (37) simultaneously translates the space $\boldsymbol{x}$ and the field $\boldsymbol{X}_{sp}$ by the same amount.

For the KD system, the weak EL equation for $\boldsymbol{X}_{sp}$ is

$$\begin{aligned} E_{\boldsymbol{X}_{sp}}(L_{KD}) &\equiv \frac{\partial L_{KD}}{\partial \boldsymbol{X}_{sp}} - \frac{D}{Dt}\frac{\partial L_{KD}}{\partial \dot{\boldsymbol{X}}_{sp}} \\ &= \frac{\partial}{\partial \boldsymbol{x}}\left[\left(-\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_{sp} + \phi - \frac{1}{2}\dot{\boldsymbol{X}}_{sp}^2\right)\delta_2\right] + \frac{\partial}{\partial \boldsymbol{x}}\cdot\left[\dot{\boldsymbol{X}}_{sp}\left(\dot{\boldsymbol{X}}_{sp} + \boldsymbol{A}\right)\delta_2\right]. \end{aligned} \tag{44}$$

Energy conservation follows from the infinitesimal criteria (29) for the symmetry transformation $(\tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\phi}, \tilde{\boldsymbol{A}}, \tilde{\boldsymbol{X}}_p) = (\boldsymbol{x}, t+\epsilon, \phi, \boldsymbol{A}, \boldsymbol{X}_{sp})$ after the weak EL equation (44) for $\boldsymbol{X}_{sp}$ and the EL equations for ϕ and $\boldsymbol{A}$ are applied, i.e.,

$$\frac{\partial}{\partial t}\left[\frac{(\nabla\phi)^2 + \boldsymbol{B}^2}{8\pi} + \sum_s \int F_s \frac{m_s \boldsymbol{v}^2}{2} d^3\boldsymbol{v}\right] + \frac{\partial}{\partial \boldsymbol{x}}\cdot\left(\sum_s \int F_s \frac{m_s \boldsymbol{v}^2}{2}\boldsymbol{v} d^3\boldsymbol{v} + \frac{\phi_{,t}\boldsymbol{A}_{,t} + \boldsymbol{E}\times\boldsymbol{B}}{4\pi}\right) = 0.$$

Similarly, the infinitesimal criteria for the symmetry group $(\tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\phi}, \tilde{\boldsymbol{A}}, \tilde{\boldsymbol{X}}_{sp}) = (\boldsymbol{x}+\epsilon\boldsymbol{u}, t, \phi, \boldsymbol{A}, \boldsymbol{X}_{sp}+\epsilon\boldsymbol{u})$ gives the momentum conservation relation

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\sum_s m_s \int F_s \boldsymbol{v} d^3\boldsymbol{v} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{4\pi}\right) \\ &+ \frac{\partial}{\partial \boldsymbol{x}}\cdot\left[\sum_s m_s \int F_s \boldsymbol{v}\boldsymbol{v} d^3\boldsymbol{v} + \frac{(\nabla\phi)^2 + \boldsymbol{B}^2 + 2\nabla\phi\cdot\boldsymbol{A}_{,t}}{8\pi}\boldsymbol{I} - \frac{\boldsymbol{E}\boldsymbol{E} + \boldsymbol{B}\boldsymbol{B} - \boldsymbol{A}_{,t}\boldsymbol{A}_{,t}}{4\pi}\right] = 0. \end{aligned} \tag{45}$$

For the KM system, the weak EL equation for $\boldsymbol{X}_{sp}$ is the same as Eq. (44). The symmetry groups $(\tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\phi}, \tilde{\boldsymbol{A}}, \tilde{\boldsymbol{X}}_{sp}) = (\boldsymbol{x}, t+\epsilon, \phi, \boldsymbol{A}, \boldsymbol{X}_{sp})$ and $(\tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\phi}, \tilde{\boldsymbol{A}}, \tilde{\boldsymbol{X}}_{sp}) = (\boldsymbol{x}+\epsilon\boldsymbol{u}, t, \phi, \boldsymbol{A}, \boldsymbol{X}_{sp}+\epsilon\boldsymbol{u})$ gives the energy and momentum conservation laws (10) and (11) after the weak EL equation for $\boldsymbol{X}_{sp}$ and EL equations for ϕ and $\boldsymbol{A}$ are applied.

In summary, a closer examination of the field theory for classical particle-field systems reveals that the particle field $\boldsymbol{X}_{sp}$ and the electromagnetic field reside on different manifolds.

This unique feature indicates that $E_{\boldsymbol{X}_{sp}}(L)$, the Euler derivative of the Lagrangian density L with respect to particle's trajectory $\boldsymbol{X}_{sp}$, does not vanish on the space-time manifold, which is surprisingly different from the standard field theory. In fact,

$$E_{\boldsymbol{X}_{sp}}(L) \equiv \frac{\partial L}{\partial \boldsymbol{X}_{sp}} - \frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{X}}_{sp}}\right) = \frac{\partial}{\partial \boldsymbol{x}} \cdot \boldsymbol{T}, \tag{46}$$

for some non-vanishing tensor $\boldsymbol{T}$. Equation (46) is what we call the weak Euler-Lagrange equation, and it is the most essential component in establishing the connection between energy-momentum conservation and space-time symmetry for classical particle-field systems. In fact, the energy-momentum conservation law follows from the infinitesimal criteria of the space-time system, after the weak Euler-Lagrange equation is applied. For the Klimontovich-Maxwell (or Vlasov-Maxwell) system, this theoretical construction explicitly links the well-known energy-momentum conservation law with the space-time symmetry, which was only cautiously suggested previously. For the reduced systems, such as the Klimontovich-Poisson (or Vlasov-Poisson) system and the Klimontovich-Darwin (Vlasov-Darwin) system, this theoretical construction enable us to start from fundamental symmetry properties in order to systematically derive the energy-momentum conservation laws, which are difficult to find otherwise.

This research was supported by the CAS Program for Interdisciplinary Collaboration Team, the JSPS-NRF-NSFC A3 Foresight Program in the field of Plasma Physics (NSFC-11261140328), and U.S. Department of Energy (DE-AC02-09CH11466).

[1] R. C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, Singapore, 2001), pp. 23–49.

[2] A. N. Kaufman and P. S. Rostler, Phys. Fluids **14**, 446 (1971).

[3] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1999), pp. 596–598, 3rd ed.

[4] H. Qin, R. C. Davidson, W. W. Lee, and R. Kolesnikov, Nuclear Instruments and Methods in Physics Research Section A **464**, 477 (2001).

[5] T. B. Krause, A. Apte, and P. J. Morrison, Phys. Plasmas **14**, 102112 (2007).

[6] L. D. Landau and E. M. Lifshitz, *The classical theory of fields* (Butterworth-Heinemann, Oxford, 1975), pp. 66–88.

[7] B. Scott and J. Smirnov, Phys. Plasmas **17**, 112302 (2010).

[8] A. J. Brizard and N. Tronko, Phys. Plasmas **18**, 082307 (2011).

[9] H. Sugama, T.-H. Watanabe, and M. Nunami, Phys. Plasmas **20**, 024503 (2013).

[10] H. Sugama, T.-H. Watanabe, and M. Nunami, Phys. Plasmas **21**, 012515 (2014).

[11] E. Noether, Nachr. König. Gesell. Wiss. Göttingen, Math.-Phys. Kl. (1918).

[12] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York, 1993), pp. 242–283.

[13] F. E. Low, Proc. R. Soc. London, Ser. A **248**, 282 (1958).

[14] P. J. Morrison, Physics Letters **80A**, 383 (1980).

[15] H. Ye and P. J. Morrison, Physics of fluids B **4**, 771 (1992).

[16] A. J. Brizard, Physical Review Letters **84**, 5768 (2000).

[17] J. Squire, H. Qin, W. M. Tang, and C. Chandre, Phys. Plasmas **20**, 022501 (2013).

[18] P. L. Similon, Ph.D. thesis, Princeton University (1981).

[19] V. K. Decyk, Physics of Fluids **25**, 1205 (1982).

The Princeton Plasma Physics Laboratory is operated
by Princeton University under contract
with the U.S. Department of Energy.

Information Services
Princeton Plasma Physics Laboratory
P.O. Box 451
Princeton, NJ 08543

Phone: 609-243-2245
Fax: 609-243-2751
e-mail: pppl_info@pppl.gov
Internet Address: http://www.pppl.gov